THEOREM (PROPERTIES OF THE WEAK TOPOLOGY)

① $\sigma(X, X^*)$ IS A HAUSDORFF TOPOLOGY

② $\sigma(X, X^*)$ ARE <u>NOT</u> METRIZABLE IF $\dim X = +\infty$

(EVEN WHEN $x_n \rightharpoonup x$ IS EQUIVALENT TO $x_n \to x$, SUCH AS ℓ_1)

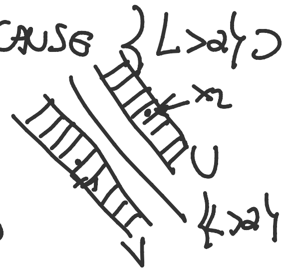
③ $\sigma(X, X^*)$ IS <u>LOCALLY</u> METRIZABLE IF X^* IS SEPARABLE ($\exists$ SUBSET DENSE AND COUNTABLE)
(ON BOUNDED SETS)

PROOF

① FIX $x_1, x_2 \in X$, $x_1 \neq x_2$ AND STRICTLY SEPARE $\{x_1\}, \{x_2\}$ USING II GEOMETRIC FORM OF HAHN-BANACH. $\exists L \in X^*, \alpha \in \mathbb{R}$ SUCH THAT $Lx_1 < \alpha < Lx_2$

$\Rightarrow x_2 \in \{L > \alpha\} \not\ni x_1$ AND IT IS OPEN IN $\sigma(X, X^*)$ BECAUSE $\{L > \alpha\} \supset U$

$U = \{x \in X : |Lx - Lx_0| < \varepsilon\}$ IF $Lx_0 > \frac{\alpha}{2}$, $\varepsilon = \frac{\alpha}{2}$



$\Rightarrow \{L > \alpha\}$ IS A NEIGHBORHOOD OF x_2
$\{L < \alpha\}$ IS A NEIGHBORHOOD OF x_1
AND THEY ARE DISJOINTED

② BY CONTRADICTION, ASSUME $\sigma(X, X^*)$ IS INDUCED BY A DISTANCE $\tilde{d}(x, y)$
IN PARTICULAR $\tilde{B}_{\frac{1}{n}}(0) = \{x \in X : \tilde{d}(x, 0) < \frac{1}{n}\}$ $\forall n \in \mathbb{N}$ IS OPEN IN $\sigma(X, X^*)$,
THAT IS $\tilde{B}_{\frac{1}{n}} \supset U_n = \{|L_1 x| < \varepsilon, \ldots, |L_N x| < \varepsilon\}$ NEIGHB. OF 0, FOR SOME $L_1, \ldots, L_N \in X^*$, $\varepsilon > 0$
IF $\dim +\infty$, $\ker L_1 \cap \ldots \cap \ker L_N \neq \emptyset$, BECAUSE OTHERWISE $A : X \to \mathbb{R}^N$
$x \xrightarrow{A} (L_1 x, \ldots, L_N x)$ IS INJECTIVE, WHICH WOULD IMPLY $\dim X \leq N$

$\exists x_n \in \ker L_1 \cap \ldots \cap \ker L_N$, SINCE IT IS A VECTOR SPACE, I CAN ASSUME $\|x_n\| \geq n$, BUT I HAVE $x_n \in U_n \subset \tilde{B}_{\frac{1}{n}}$ SO $d(x_n, 0) < \frac{1}{n} \xrightarrow[n \to \infty]{} 0$,
THAT IS $x_n \to 0$, IN CONTRADICTION WITH x_n UNBOUNDED.

③ ASSUME $\exists \{L_n\}_{n \in \mathbb{N}}$ DENSE IN THE CLOSED UNIT BALL OF X^*
$\overline{B_1(0)} := \{\|L\|_{X^*} \leq 1\}$ AND WE SET $\tilde{d}(x, y) := \sum_{n=1}^{+\infty} \frac{|L_n(x-y)|}{2^n}$
LET US SHOW THAT $\tilde{d}$ INDUCES THE

$\dots \} \|L\|_{*} \leq 1\}$ AND WE SET $\tilde{d}(x,y) := \sum_{k=1} \frac{|L_k(x-y)|}{2^k}$

LET US SHOW THAT $\tilde{d}$ INDUCES THE WEAK TOPOLOGY ON BOUNDED SETS, IT IS NOT RESTRICTIVE TO ASSUME $\|x\| \leq 1$

GIVEN A WEAK NEIGHB. $U = U_{M_1, \dots, M_N, \varepsilon}(x_0)$ WE SHOW $\tilde{B}_r(x_0) \subset U$

WE MAY ASSUME $\|M_i\| \leq 1$ $\forall i = 1, \dots, N$ (OTHERWISE I RESCALE ε)

$\exists L_{k_1}, \dots, L_{k_N}$ SUCH THAT $\|L_{k_i} - M_i\| \leq \frac{\varepsilon}{4}$, THEN I CAN TAKE $r < \frac{\varepsilon}{2^{k_i+1}}$

IN FACT, IF $x \in \tilde{B}_r(x_0)$, WE HAVE

$$|M_i(x - x_0)| \leq |M_i(x-x_0) - L_{k_i}(x-x_0)| + |L_{k_i}(x-x_0)|$$
$$\leq \|M_i - L_{k_i}\| \|x - x_0\| + 2^{k_i} \tilde{d}(x, x_0)$$
$$\leq \frac{\varepsilon}{4}(\|x\| + \|x_0\|) + 2^{k_i} r \leq \frac{\varepsilon}{4} \cdot 2 + \frac{\varepsilon}{2} = \varepsilon \Rightarrow x \in U$$

NOW, GIVEN $\tilde{B}_r(x_0)$, WE FIND A WEAK NEIGHBORHOOD $U \subset \tilde{B}_r(x_0)$

U WILL BE $U = U_{M_1, \dots, M_N, \varepsilon}$, I CHOOSE $M_i = L_i$ N SUCH THAT $\frac{1}{2^{N-1}} < \frac{r}{2}$

$\varepsilon = \frac{r}{2}$. SO IF $\underset{\|x\| \leq 1}{x \in U}$, $\tilde{d}(x, x_0) = \sum_{k=1}^{N} \frac{|L_k(x-x_0)|}{2^k} + \sum_{k=N+1}^{+\infty} \frac{|L_k(x-x_0)|}{2^k}$

$$\underset{(x \in U)}{\leq} \frac{r}{2} \sum_{k=1}^{\infty} \frac{1}{2^k} + \sum_{k=N+1}^{+\infty} \frac{\|L_k\| (\|x\| + \|x_0\|)}{2^k}$$
$$\leq \frac{r}{2} + \sum_{k=N+1}^{+\infty} \frac{1}{2^{k-1}}$$
$$< \frac{r}{2} + \frac{r}{2} = r \Rightarrow x \in \tilde{B}_r(x_0)$$

THEOREM (CHARACTERIZATION OF CLOSED CONVEX SETS)

LET X BE A NORMED SPACE AND $K \subset X$ A CONVEX SUBSET

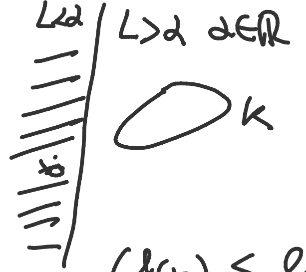
THEN K IS CLOSED $\Leftrightarrow$ K IS WEAKLY CONVEX.

PROOF: IF K IS WEAKLY ~~CONVEX~~ CLOSED THEN K IS ALWAYS CLOSED.

NOW, SUPPOSE K IS CLOSED AND CONVEX, WE WILL SHOW $X \setminus K$ IS WEAKLY OPEN: FIX $x_0 \notin K$, WE STRICTLY SEPARATE $\{x_0\}$ K USING

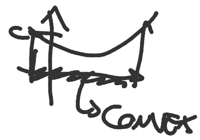
..., ... IS CLOSED AND ..., WE WILL SHOW $X \setminus K$ IS WEAKLY OPEN: FIX $x_0 \notin K$, WE STRICTLY SEPARATE $\{x_0\}$, K USING THE II GEOM. FORM OF HAHN-BANACH, THAT IS $\exists L \in X^*$ SUCH THAT $Lx_0 < \alpha < Lx \quad \forall x \in K$. $\Rightarrow x_0 \in \{L < \alpha\} \subset X \setminus K$

$\{L<\alpha\}$ IS A WEAK NEIGHBORHOOD OF x_0 CONTAINED IN $X \setminus K$, SO $X \setminus K$ IS WEAKLY OPEN.



$\boxed{\text{COROLLARY}}$ A CONVEX FUNCTION $f: X \to \mathbb{R}$ IS LOWER SEMI-CONTINUOUS IFF f IS LOWER SEMICONTINUOUS WITH THE WEAK TOPOLOGY $\left(f(x_0) \le \liminf_{x \to x_0} f(x)\right)$

PROOF f IS CONVEX $\Leftrightarrow$ ALL SUBLEVELS $f^{-1}((-\infty, c])$ ARE CONVEX



SINCE $f^{-1}((-\infty, c])$ IS CONVEX, IT IS CLOSED IFF IT IS WEAKLY CLOSED

$\Updownarrow$ f IS LOWER SEMI-CONT. $\qquad$ $\Updownarrow$ f IS WEAKLY LOWER SEMI-CONTINUOUS

$\boxed{\text{REMARK}}$ IN THE CASE OF $f(x) = \|x\|$, I GET $\|x\| \le \liminf_{n \to \infty} \|x_n\|$ IF $x_n \rightharpoonup x$. THE INEQUALITY MAY BE STRICT: WE SAW $x_n = e_n \in \ell_2$ $\|e_n\| = 1$, $e_n \rightharpoonup 0$. THE NORM IS NOT WEAKLY CONTINUOUS

WE WANT TO WEAKEN EVEN MORE THE WEAK ~~TOPOLOGY~~ CONVERGENCE, BY TESTING CONVERGENCE ONLY ON SOME ELEMENTS OF THE DUAL. ON X^*, THERE IS A SPECIAL SUBSET OF X^{**} (THE ONES COMING FROM X)

$\boxed{\text{DEF}}$ LET X^* BE THE DUAL OF THE NORMED SPACE X, $L \in X^*$ AND $\{L_n\}$ A SEQ IN X^*. WE SAY L_n CONVERGES WEAKLY* TO L IF $L_n x \xrightarrow[n \to \infty]{} Lx \quad \forall x \in X$. WE USE THE SYMBOL $L_n \xrightarrow{*} L$.

$\boxed{\text{REMARKS}}$ ① IF $L_n \rightharpoonup L$ THEN $L_n \overset{*}{\rightharpoonup} L$ ② IF $L_n \overset{*}{\rightharpoonup} L$ THEN $\{L_n\}$ IS BOUNDED

③ IF X IS REFLEXIVE, THEN $L_n \rightharpoonup L \Leftrightarrow L_n \overset{*}{\rightharpoonup} L$.

$\boxed{\text{DEF}}$ WE DEFINE THE WEAK* TOPOLOGY ON X^* AS THE ONE GIVEN BY THE FUNDAMENTAL NEIGHBORHOODS $U_{x_1 \dots x_N, \varepsilon}(L_0) = \{L \in X^* : |Lx_1 - L_0 x_1| < \varepsilon, \dots, |Lx_N - L_0 x_N| < \varepsilon\}$ IT IS DENOTED WITH $\sigma(X^*, X)$

$\boxed{\text{REMARKS}}$ ① IF X IS REFLEXIVE $\sigma(X^*, X)$ COINCIDES WITH $\sigma(X^*, X^{**})$

... WITH $\sigma(X^*, X)$

$\boxed{\text{REMARKS}}$ ① IF X IS REFLEXIVE, $\sigma(X^*, X)$ COINCIDES WITH $\sigma(X^*, X^{**})$

② $L_n \xrightarrow{*} L \iff L_n$ CONVERGES TO L IN $\sigma(X^*, X)$.

③ ANY $\Lambda \in X^{**}$ OF THE KIND $\Lambda: L \to Lx$ IS CONTINUOUS IN $\sigma(X^*, X)$ AND $\sigma(X^*, X)$ IS THE LESS FINE SUCH THAT IT IS TRUE

④ $\sigma(X^*, X)$ IS HAUSDORFF, NOT METRIZABLE, LOCALLY METRIZABLE IF $\underline{X}$ IS SEPARABLE

$\boxed{\text{PROP}}$ THE NORM $\|\cdot\|: X \to \mathbb{R}$ IS WEAKLY* LOWER-SEMICONTINUOUS SEQUENTIALLY, THAT IS IF $L_n \xrightarrow{*} L$, THEN $\|L\| \le \liminf_{n \to \infty} \|L_n\|$.

PROOF ASSUME $L_n \xrightarrow{*} L$, TAKE $x \in X$ WITH $\|x\| \le 1$. THEN

$$\underbrace{L_n x}_{\downarrow\ Lx} \le \|L_n\| \|x\| \le \|L_n\|, \quad \text{PASS TO THE } \liminf_{n\to\infty} \Rightarrow Lx \le \liminf_{n\to\infty} \|L_n\|$$

PASS TO $\sup_{\|x\| \le 1} \Rightarrow \|L\| \le \liminf_{n \to \infty} \|L_n\|$

$\boxed{\text{EXAMPLE}}$ $X = c_0$, $L_n: x \to x(n)$ (CORRESPONDING TO $e_n \in \ell_1$)

$L_n \xrightarrow{*} 0$ BUT $L_n \not\rightharpoonup 0$ BECAUSE $\Lambda: L \to \sum_{k=1}^{\infty} L e_k$ $\quad \Lambda \in X^{**}$ $\left(\Lambda: y \to \sum_{k=1}^{+\infty} y(k)\right)$

$\Lambda L = 1 \not\to 0$