$W_0^{1,p}((a,b)) := \{u \in W^{1,p}((a,b)) : u(a)=u(b)=0\}$ $\quad a,b \in \mathbb{R}$

<u>POINCARÉ INEQUALITY</u> $u \in W_0^{1,p} \Rightarrow \|u'\|_{L^p} \leq \|u\|_{W^{1,p}} \leq C\|u'\|_{L^p}$

$\Downarrow$

$\|u'\|_{L^p}$ IS AN EQUIVALENT NORM

CAN WE DEFINE $W_0^{1,p}(\mathbb{R})$? WE CAN REQUIRE $\lim_{x\to\pm\infty} u(x)=0$.

WE CAN BUT $W_0^{1,p}(\mathbb{R}) = W^{1,p}(\mathbb{R})$ $\quad (p<\infty)$

$\boxed{\text{REMARK}}$ IF $u \in W^{1,p}(\mathbb{R})$, THEN $u(x) \xrightarrow[x\to\pm\infty]{} 0$

IF $u \in L^p(\mathbb{R})$, IT IS FALSE

IT IS TRUE ON A SEQUENCE $x_n \to +\infty$ SUCH THAT $u(x_n) \xrightarrow[n\to\infty]{} 0$

ASSUME $u' \in L^p$ AND, BY CONTRADICTION, $\exists y_n \to +\infty$ SUCH THAT $u(y_n) \geq \delta > 0$

APPLY THE FUND. THM CALCULUS TO $|u|^{p-1}u$

$$\underbrace{|u(y_n)|^{p-1}u(y_n)}_{\geq \delta^p} - \underset{\substack{\downarrow\\0}}{|u(x_n)|^{p-1}u(x_n)} = \int_{x_n}^{y_n} p u'|u|^{p-1} = p\int_{\mathbb{R}} \underline{u'|u|^{p-1}}\, \chi_{[x_n,y_n]} \longrightarrow 0$$

WE APPLY THE DOMINATED CONVERGENCE THEOREM: $u'|u|^{p-1} \in L^1$

BECAUSE $\int |u|^{p-1}u' \underset{\substack{\uparrow\\ \text{HÖLDER}}}{\leq} \left(\int|u|^p\right)^{1-\frac{1}{p}}\left(\int|u'|^p\right)^{\frac{1}{p}} < +\infty$

<u>WE GOT A CONTRADICTION</u>

$\boxed{\text{LEMMA}}$ $C_0^\infty((a,b))$ IS DENSE IN $W_0^{1,p}((a,b))$ IF $p<+\infty$

$\boxed{\text{REMARK}}$ ① $C_0^\infty((a,b))$ IS NOT DENSE IN $W^{1,p}((a,b))$ BECAUSE IF $u(a)\neq 0$, u CANNOT BE APPROXIMATED BY COMPACTLY-SUPPORTED FUNCTIONS: IF $\varphi \in C_0^\infty((a,b))$, BY SOBOLEV EMBEDDING THEOREM,

$$\underset{\substack{\vee\!/\\|u(a)|\neq 0}}{\|\varphi-u\|_{L^\infty}} \leq \|\varphi-u\|_{W^{1,p}}$$

② $C_0^\infty((a,b))$ IS NOT DENSE IN $W_0^{1,\infty}((a,b))$ BECAUSE $\|\varphi'-u'\|_{L^\infty} \leq \|\varphi-u\|_{W^{1,\infty}}$ AND u' CANNOT BE APPROXIMATED IN L^∞ BY $\varphi \in C^\infty$ IF u' IS NOT CONTINUOUS.

<u>PROOF</u> TAKE $u \in W_0^{1,p}((a,b))$. SINCE $u' \in L^p$, $\exists \{\psi_n\}$ SEQ. IN $C_0^\infty((a,b))$

CONTINUOUS.

<u>PROOF</u> TAKE $u \in W_0^{1,p}((a,b))$. SINCE $u' \in L^p$, $\exists \{\psi_n\}$ SEQ. IN $C_c^\infty((a,b))$ SUCH THAT $\psi_n \to u'$ IN $L^p((a,b))$. DEFINE $\varphi_n(x) := \int_a^x \psi_n - \left(\int_a^b \psi_n\right) \eta_0(x)$

FOR SOME $\eta_0 \in C^\infty((a,b))$ SUCH THAT $\eta_0 \equiv 1$ CLOSE TO b

$\eta_0 \equiv 0$ CLOSE TO a

X NEAR $a \Rightarrow \int_a^x \psi_n = 0$, $\eta_0(x) = 0 \Rightarrow \varphi_n(x) = 0$

X NEAR $b \Rightarrow \int_a^x \psi_n = \int_a^b \psi_n$, $\eta_0 = 1 \Rightarrow \varphi_n(x) = 0$ ✓

LET US SHOW THAT $\varphi_n \to u$ IN $W^{1,p}$

KEY: $\int_a^b u' = u(b) - u(a) = 0$.

$$\|\varphi_n - u\|_{W^{1,p}} \overset{\text{POINCARÉ INEQ.}}{\leq} C \|\varphi_n' - u'\|_{L^p} = C \left\| \psi_n(x) - \int_a^b (\psi_n - u') \eta_0'(x) - u'(x) \right\|_{L^p}$$

$$\leq \underbrace{C\|\psi_n - u\|_{L^p}}_{\downarrow \, 0} + \underbrace{C \int_a^b |\psi_n - u'|}_{\leq C \|\psi_n - u'\|_{L^p} \to 0}$$

SIMILARLY, $C_0^\infty(\mathbb{R})$ IS DENSE IN $W^{1,p}(\mathbb{R})$.

$\boxed{\text{COR}}$ ① IF $u \in W^{1,p}((a,b))$ THEN $\int_a^b u v' = -\int_a^b u' v$ ALSO FOR $v \in W_0^{1,1}((a,b))$

② IF $u, v \in W^{1,p}((a,b))$, THEN $uv \in W^{1,p}((a,b))$. IF $|u| \geq \delta > 0$, THEN $\frac{1}{u} \in W^{1,p}((a,b))$

IF $H \in \text{LIP}((a,b))$ THEN $H(u)$ IS IN $W^{1,p}((a,b))$ $a, b \in \mathbb{R}$

a, b CAN BE ±∞

$(uv)' = uv' + u'v$ $\quad \frac{1}{u}' = -\frac{u'}{u^2}$ $\quad H(u)' = H'(u) u'$

<u>PROOF</u> ① FIX $v \in W_0^{1,1}(a,b)$, BY THE DENSITY LEMMA $\exists \{\varphi_n\}$ $\varphi_n \to v$ IN $W^{1,1}((a,b))$.

BY DEF. OF WEAK DERIVATIVE, $\int_a^b u' \varphi_n = -\int_a^b u \varphi_n'$

$$\left| \int_a^b u' \varphi_n - \int_a^b u' v \right| \leq \int |u'| |\varphi_n - v| \leq |u'|_{L^p} |\varphi_n - v|_{L^{p'}} \leq |u|_{W^{1,p}} |\varphi_n - v|_{W^{1,1}} \to 0$$

$$\left| \int_a^b u \varphi_n' - \int_a^b u v' \right| \leq \int |u| |\varphi_n' - v'| \leq |u|_{L^\infty} |\varphi_n' - v'|_{L^1} \leq |u|_{W^{1,p}} |\varphi_n - v|_{W^{1,1}} \to 0$$

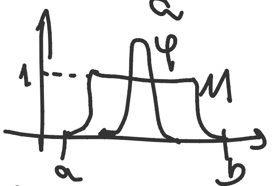
$$\Rightarrow \underline{\int u' v} \leftarrow \int_a^b u' \varphi_n = -\int_a^b u \varphi_n' \to -\int_a^b u \underline{v'}$$

$\Rightarrow \underline{\int uv} \leftarrow \int_a u\varphi_n = -\int_a u\varphi_n' \longrightarrow -\int_a \underline{uv'}$

② FIX $\varphi \in C_0^1((a,b))$ AND WE WANT TO SHOW $\int_a^b uv\varphi' = -\int_a^b \varphi(uv'+u'v)$

TAKE $\eta_0 \in C_c^\infty((a,b))$ SUCH THAT $\eta_0 \equiv 1$ ON $supp(\varphi)$

η_0 IS A "CUTOFF" $\Rightarrow \eta_0\varphi = \varphi$



$\eta_0 u \in W_0^{1,p}((a,b)) \Rightarrow \exists \psi_n \longrightarrow \eta_0 u$ IN $W^{1,p}$

$\psi_n \in C_0^\infty((a,b))$ BY DENSITY LEMMA

THEN $\psi_n\varphi \longrightarrow \underline{u\varphi}$ IN L^p BECAUSE

$$\int_a^b |\psi_n\varphi - u\varphi|^p = \int_a^b |\psi_n\varphi - \eta_0 u\varphi|^p \leq C\int_a |\psi_n - \eta_0 u|^p \longrightarrow 0$$

SIMILARLY, $\psi_n'\varphi \longrightarrow \underline{u'\varphi}$ IN L^p AND $\psi_n\varphi' \longrightarrow \underline{u\varphi'}$

BY DEF. OF WEAK DERIVATIVE $\int v'\varphi\psi_n = -\int v(\varphi\psi_n)' = -\int v\varphi'\psi_n - \int v\varphi\psi_n'$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$\Rightarrow \int uv\varphi' = -\int \varphi(uv'+u'v)$ $\qquad \int v'\varphi u \qquad -\int v\varphi' u \quad -\int v\varphi u'$

THAT IS uv HAS A WEAK DERIVATIVE $uv'+u'v$

$u \in L^\infty$ $v' \in L^p \Rightarrow uv' \in L^p$

$u' \in L^p$ $v \in L^\infty \Rightarrow u'v \in L^p$

$\Rightarrow uv$ IS IN $W^{1,p}$

CASE $p=+\infty$: $u,v \in W^{1,\infty} \Rightarrow u,v \in W^{1,p}_{loc}$, IN PARTICULAR $u,v \in W^{1,p}$ ON $supp(\varphi) \Rightarrow$ AS BEFORE $\int uv\varphi' = -\int\varphi(uv'+u'v)$

$\Rightarrow uv$ HAS A WEAK DER. $uv'+u'v$. $u,v,u',v' \in L^\infty \Rightarrow (uv)' \in L^\infty$

$\underline{\text{WEAK SOLUTIONS OF DIFFERENTIAL EQUATIONS}}$

$\boxed{\text{DEF}}$ A WEAK SOLUTION OF THE DIFFERENTIAL EQUATION

$(\ast\ast)\begin{cases} (-p(x)u')' + q(x)u = f & \text{ON } (a,b) \\ u(a) = u(b) = 0 \end{cases}$ IS A FUNCTION $\underline{u \in W_0^{1,2}((a,b))}$

SUCH THAT $-p(x)u'$ HAS A WEAK DERIVATIVE SATISFYING $(-p(x)u')' = f - q(x)u$ a.e. ON (a,b), THAT IS

$$(\ast) \int_a^b \left(p(x)u'\varphi' + q(x)u\varphi\right) = \int_a^b f\varphi \quad \forall \varphi \in C_0^1((a,b))$$

$f, q \in L^1((a,b))$ $\quad p \in L^\infty((a,b))$ $\quad p \geq \delta > 0$ $\quad q \geq 0$ $\quad$ a.e. ON (a,b)

(TIPICALLY, $p \equiv 1$, $q \equiv 0$)

$-u'' = f$ $\quad \int_a^b u'\varphi' = \int_a^b f\varphi$

$\boxed{\text{REMARK}}$ ① IF u IS A WEAK SOL. THEN $\circledast$ IS TRUE ALSO FOR $v \in W_0^{1,2}((a,b))$

TAKE $\varphi_n \in C_c^\infty$ $\varphi_n \longrightarrow v$ IN $W^{1,2}$ $\Rightarrow$

① IF u IS A WEAK SOL. THEN ⊛ IS TRUE ALSO FOR $v \in W_0^{1,2}(a,b)$

TAKE $\varphi_n \in C_c^\infty$ $\varphi_n \to v$ IN $W_0^{1,2}$ $\Rightarrow \int \big(p(x) u' \underset{\substack{\downarrow \\ \int p(x) u' v' + quv}}{\varphi_n'} + q(x) u \varphi_n\big) = \int f \varphi_n \to \int f v$

FOR ANY v, $(u,v) = Lv$

② IF $p \in C^1([a,b])$, $q, f \in C([a,b])$ THEN INTEGRATING BY PARTS YOU CAN SEE THAT WEAK SOLUTIONS ARE CLASSICAL SOLUTIONS

③ IN GENERAL, SOLUTIONS ARE NOT $C^2([a,b])$ $\begin{cases} -u'' = \operatorname{sign}(x) \\ u(1) = u(-1) = 0 \end{cases}$ $\Rightarrow u(x) = \dfrac{x(1-|x|)}{2}$, C^1 NOT C^2

④ IF WE HAVE $\begin{cases} (-pu')' + qu = f \\ u(a) = \alpha \\ u(b) = \beta \end{cases}$ FOR GENERIC α, β

THEN WE MAY WRITE $u = v_0 + w$ WITH $v_0(x) = Ax + B$ SUCH THAT $v_0(a) = \alpha$, $v_0(b) = \beta$ AND w SOLVING $\begin{cases} (-pw)' + qw' = \tilde{f} \\ w(a) = w(b) = 0 \end{cases}$

$\boxed{\text{PROP}}$ LET u BE A WEAK SOL. OF ⊛⊛ WITH $p \in L^\infty$ $q, f \in L^1$

THEN, $u \in W_0^{1,\infty}((a,b))$

IF $p \in W^{1,p}((a,b))$ AND $q, f \in L^p((a,b))$ FOR $p \in [1, +\infty]$ THEN $u' \in W^{1,p}$ $(\Rightarrow u \in C^1)$

IF $p \in C^{k+1}((a,b))$ AND $q, f \in C^k((a,b))$ FOR $k \in \mathbb{N}$ THEN $u \in C^{k+2}$

$\boxed{\text{PROOF}}$ BY DEF. pu' HAS A WEAK DER. $qu - f$. $\underset{u \in L^\infty}{q \in L^1}$ $f \in L^1$ $\Rightarrow \underset{L^1}{\underset{\in}{qu-f}}$

$pu' \in W^{1,1} \subset L^\infty$ $\Rightarrow u' = \underset{\in L^\infty}{\left(\frac{1}{p}\right)} pu' \in L^\infty$ BECAUSE $p \geq \delta > 0$

ASSUME $p \in W^{1,p}$ $q, f \in L^p \Rightarrow qu - f \in L^p \Rightarrow \underline{pu' \in W^{1,p}}$.

BY THE COROLLARY, $\underline{\frac{1}{p} \in W^{1,p}}$ AND $u' = \frac{1}{p} \cdot pu' \in W^{1,p}$ SINCE IT IS A PRODUCT OF FUNCTIONS IN $W^{1,p}$.