Università degli Studi Roma Tre - Corso di Laurea in Matematica

Esercitazione di AM120

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Argomento: Formula di Taylor

1. Determinare lo sviluppo di Taylor all'ordine 4 di $f(x) = 2e^{-x^2} - \cos(2x) - 1$ e calcolare il limite $\lim_{x\to 0} \frac{f(x)}{x^4}$.

   Dagli sviluppi di Taylor

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2} + o\left(x^2\right) \\ \cos x &= 1 - \frac{x^2}{2} + \frac{x^4}{24} \end{aligned}$$

   otteniamo

$$f(x) = 2\left(1 - x^2 + \frac{x^4}{2} + o\left(x^4\right)\right) - \left(1 - 2x^2 + \frac{2}{3}x^4 + o\left(16x^4\right)\right) - 1 = \frac{x^4}{3} + o\left(x^4\right),$$

   quindi il limite vale

$$\lim_{x\to 0} \frac{f(x)}{x^4} = \lim_{x\to 0} \frac{\frac{x^4}{3} + o\left(x^4\right)}{x^4} = \lim_{x\to 0} \left(\frac{1}{3} + \frac{o\left(x^4\right)}{x^4}\right) = \frac{1}{3}.$$

2. Determinare lo sviluppo di Taylor all'ordine 3 di $f(x) = \cosh x - \frac{e^x}{1+x}$ e calcolare il limite $\lim_{x\to 0} \frac{f(x)}{x^3}$.

   Dai noti sviluppi di Taylor

$$\begin{aligned} \cosh x &= 1 + \frac{x^2}{2} + o\left(x^3\right) \\ e^x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o\left(x^3\right) \\ \frac{1}{1-x} &= 1 + x + x^2 + x^3 + o\left(x^3\right) \end{aligned}$$

   si ottiene

$$\begin{aligned} \cosh x - \frac{e^x}{1+x} &= 1 + \frac{x^2}{2} + o\left(x^3\right) - \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + o\left(x^3\right)\right)\left(1 - x + x^2 - x^3 + o\left(x^3\right)\right) \\ &= 1 + \frac{x^2}{2} + o\left(x^3\right) - \left(\left(1 - x + x^2 - x^3 + o\left(x^3\right)\right) + x\left(1 - x + x^2 + o\left(x^2\right)\right) + \right. \\ &+ \left. \frac{x^2}{2}(1 - x + o(x)) + \frac{x^3}{6}(1 + o(1)) + o\left(x^3\right)\right) \\ &= 1 + \frac{x^2}{2} + o\left(x^3\right) - \left(1 - x + x^2 - x^3 + x - x^2 + x^3 + \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^3}{6} + o\left(x^3\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= 1+\frac{x^2}{2}+o\left(x^3\right)-\left(1+\frac{x^2}{2}-\frac{x^3}{6}+o\left(x^3\right)\right) \\
&= \frac{x^3}{3}+o\left(x^3\right),
\end{aligned}
$$

dunque il limite vale

$$\lim_{x\to 0}\frac{f(x)}{x^3}=\lim_{x\to 0}\frac{\frac{x^3}{3}+o\left(x^3\right)}{x^3}=\lim_{x\to 0}\left(\frac{1}{3}+\frac{o\left(x^3\right)}{x^3}\right)=\frac{1}{3}.$$

3. Determinare lo sviluppo di Taylor all'ordine 2 di $f(x)=e^{\sin x}-\sqrt{1+2x}$ e $g(x)=\tan^2 x$ e calcolare il limite $\lim_{x\to 0}\frac{f(x)}{g(x)}$.

   Dagli sviluppi di Taylor

$$
\begin{aligned}
e^x &= 1+x+\frac{x^2}{2}+o\left(x^2\right) \\
\sin x &= x+o\left(x^2\right) \\
\sqrt{1+x} &= 1+\frac{x}{2}-\frac{x^2}{8}+o\left(x^2\right) \\
\tan x &= x+o\left(x^2\right)
\end{aligned}
$$

   otteniamo

$$
\begin{aligned}
f(x) &= e^{x+o\left(x^2\right)}-\left(1+x-\frac{x^2}{2}+o\left(4x^2\right)\right) \\
&= 1+x+o\left(x^2\right)+\frac{\left(x+o\left(x^2\right)\right)^2}{2}+o\left(x+o\left(x^2\right)\right)^2-\left(1+x-\frac{x^2}{2}+o\left(x^2\right)\right) \\
&= 1+x+o\left(x^2\right)+\frac{x^2}{2}+o\left(x^2\right)+o\left(x^2\right)-\left(1+x-\frac{x^2}{2}+o\left(x^2\right)\right) \\
&= x^2+o\left(x^2\right)
\end{aligned}
$$

   e

$$\tan^2 x=\left(x+o\left(x^2\right)\right)^2=x^2+o\left(x^2\right),$$

   dunque

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{x^2+o\left(x^2\right)}{x^2+o\left(x^2\right)}=\lim_{x\to 0}\frac{1+\frac{o\left(x^2\right)}{x^2}}{1+\frac{o\left(x^2\right)}{x^2}}=1.$$

4. Calcolare i seguenti limiti:

   (a) $\lim_{x\to 0}\frac{x\arctan(x)+2\log(\cos x)}{\sin^2 x-\sin\left(x^2\right)}$;

   Dagli sviluppi

$$
\begin{aligned}
\arctan x &= x-\frac{x^3}{3}+o\left(x^3\right) \\
\log(1+x) &= x-\frac{x^2}{2}+o\left(x^2\right) \\
\cos x &= 1-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^4\right) \\
\sin x &= x-\frac{x^3}{6}+o\left(x^4\right)
\end{aligned}
$$

si ottiene

$$\begin{aligned}
x\arctan(x)+2\log(\cos x) &= x\left(x-\frac{x^3}{3}+o\left(x^3\right)\right)+2\log\left(1-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^4\right)\right)\\
&= x^2-\frac{x^4}{3}+o\left(x^4\right)+2\left(-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^4\right)-\frac{\left(-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^4\right)\right)^2}{2}\right.\\
&+ \left. o\left(-\frac{x^2}{2}+\frac{x^4}{24}+o\left(x^4\right)\right)^2\right)\\
&= x^2-\frac{x^4}{3}+o\left(x^4\right)-2\left(-\frac{x^2}{2}+\frac{x^4}{12}+o\left(x^4\right)\right)\\
&= -\frac{x^4}{2}+o\left(x^4\right)
\end{aligned}$$

e

$$\begin{aligned}
\sin^2 x-\sin\left(x^2\right) &= \left(x-\frac{x^3}{6}+o\left(x^3\right)\right)^2-\left(x^2+o\left(x^4\right)\right)^2\\
&= \left(x^2+\frac{x^6}{36}+o\left(x^6\right)-\frac{x^4}{3}+o\left(x^4\right)+o\left(x^6\right)\right)-\left(x^4+o\left(x^4\right)\right)\\
&= -\frac{x^4}{3}+o\left(x^4\right),
\end{aligned}$$

quindi

$$\lim_{x\to 0}\frac{x\arctan(x)+2\log(\cos x)}{\sin^2 x-\sin\left(x^2\right)}=\lim_{x\to 0}\frac{-\frac{x^4}{2}+o\left(x^4\right)}{-\frac{x^4}{3}+o\left(x^4\right)}=\lim_{x\to 0}\frac{-\frac{1}{2}+\frac{o\left(x^4\right)}{x^4}}{-\frac{1}{3}+\frac{o\left(x^4\right)}{x^4}}=\frac{3}{2}.$$

(b) $\displaystyle\lim_{x\to 0}\frac{\left(\frac{2}{2+x^2}-\cos x\right)\sin\frac{1}{x}}{\log\left(1+x^3\right)}$;

Dagli sviluppi

$$\begin{aligned}
\frac{1}{1-x} &= 1+x+o\left(x^{\frac{3}{2}}\right)\\
\cos x &= 1-\frac{x^2}{2}+o\left(x^3\right)\\
\log(1+x) &= x+o(x)
\end{aligned}$$

si ottiene

$$\begin{aligned}
\frac{2}{2+x^2}-\cos x &= \frac{1}{1+\frac{x^2}{2}}-\cos x\\
&= 1-\frac{x^2}{2}+o\left(x^3\right)-1+\frac{x^2}{2}+o\left(x^3\right)+\\
&= o\left(x^3\right)
\end{aligned}$$

e

$$\log\left(1+x^3\right)=x^3+o\left(x^3\right),$$

dunque

$$\lim_{x\to 0}\frac{\frac{2}{2+x^2}-\cos x}{\log\left(1+x^3\right)}=\lim_{x\to 0}\frac{o\left(x^3\right)}{x^3+o\left(x^3\right)}=\lim_{x\to 0}\frac{\frac{o\left(x^3\right)}{x^3}}{1+\frac{o\left(x^3\right)}{x^3}}=0$$

Poiché $\left|\sin\frac{1}{x}\right| \leq 1$, concludiamo che

$$\left|\frac{\left(\frac{2}{2+x^2}-\cos x\right)\sin\frac{1}{x}}{\log\left(1+x^3\right)}\right| \leq \left|\frac{\frac{2}{2+x^2}-\cos x}{\log\left(1+x^3\right)}\right| \underset{x\to 0}{\to} 0.$$

(c) $\lim\limits_{x\to 0}\dfrac{e^{e^x}-ee^x}{\sin(2x)-2\log(1+x)}$;

Dagli sviluppi di Taylor

$$\begin{aligned} e^x &= 1+x+\frac{x^2}{2}+o\left(x^2\right) \\ \sin x &= x+o\left(x^2\right) \\ \log(1+x) &= x-\frac{x^2}{2}+o\left(x^2\right) \end{aligned}$$

si ottiene

$$\begin{aligned} e^{e^x}-ee^x &= e\left(e^{e^x-1}-e^x\right) \\ &= e\left(1+(e^x-1)+\frac{(e^x-1)^2}{2}+o\left(x^2\right)-\left(1+x+\frac{x^2}{2}+o\left(x^2\right)\right)\right) \\ &= e\left(1+x+\frac{x^2}{2}+o\left(x^2\right)+\frac{\left(x+\frac{x^2}{2}+o\left(x^2\right)\right)^2}{2}+o\left(x^2\right)-\left(1+x+\frac{x^2}{2}+o\left(x^2\right)\right)\right) \\ &= e\left(\frac{\left(x+\frac{x^2}{2}+o\left(x^2\right)\right)^2}{2}+o\left(x^2\right)\right) \\ &= \frac{e}{2}x^2+o\left(x^2\right) \end{aligned}$$

e

$$\sin(2x)-2\log(1+x)=2x+o\left(x^2\right)-2\left(x-\frac{x^2}{2}+o\left(x^2\right)\right)=x^2+o\left(x^2\right),$$

dunque

$$\lim_{x\to 0}\frac{e^{e^x}-e^x}{\sin(2x)-2\log(1+x)}=\lim_{x\to 0}\frac{-\frac{e}{2}x^2+o\left(x^2\right)}{x^2+o\left(x^2\right)}=\lim_{x\to 0}\frac{\frac{e}{2}+\frac{o\left(x^2\right)}{x^2}}{1+\frac{o\left(x^2\right)}{x^2}}=\frac{e}{2}.$$

(d) $\lim\limits_{x\to 0}\left(\dfrac{1}{\log(1+x)}-\dfrac{1}{\arctan x}\right)$;

Riscrivendo il limite come $\lim\limits_{x\to 0}\dfrac{\arctan(x)-\log(1+x)}{\arctan(x)\log(1+x)}$, potrà essere calcolato utilizzando gli sviluppi di Taylor di numeratore e denominatore. Questi ultimi si ottengono da quelli ben noti

$$\begin{aligned} \log(1+x) &= x-\frac{x^2}{2}+o\left(x^2\right) \\ \arctan x &= x+o\left(x^2\right), \end{aligned}$$

da cui

$$\arctan x-\log(1+x)=x+o\left(x^2\right)-\left(x-\frac{x^2}{2}+o\left(x^2\right)\right)=\frac{x^2}{2}+o\left(x^2\right)$$

e

$$\log(1+x)\arctan(x) = \left(x - \frac{x^2}{2} + o\left(x^2\right)\right)\left(x + o\left(x^2\right)\right) = x^2 + o\left(x^2\right),$$

dunque abbiamo $\lim_{x\to 0} \frac{\arctan(x) - \log(1+x)}{\arctan(x)\log(1+x)} = \lim_{x\to 0} \frac{\frac{x^2}{2} + o\left(x^2\right)}{x^2 + o\left(x^2\right)} = \lim_{x\to 0} \frac{\frac{1}{2} + \frac{o\left(x^2\right)}{x^2}}{1 + \frac{o\left(x^2\right)}{x^2}} = \frac{1}{2}.$