EXERCISE 3 ON HILBERT SPACES' SHEET

$H: L^2(B_1(0))$ $B_1(0) \subset \mathbb{R}^N$ UNIT BALL

$E = \{$ radial functions $\}$ $E \triangleleft H$

1) SHOW THAT $E^\perp = \underbrace{\{ g \in H : \int_{\{|x|=r\}} g(x) = 0 \text{ FOR A.E. } r \in (0,1) \}}_{=:F}$

$F \subset E^\perp$: TAKE $f \in E$, $g \in F$

$$(f,g) = \int_0^1 \left(\int_{|x|=r} f(x) g(x)\,dx \right) dr \overset{\substack{f \text{ CONSTANT ON } |x|=r \\ \downarrow}}{=} \int_0^1 f(r) \Big(\underbrace{\int_{|x|=r} g(x)\,dx}_{=0} \Big) dr = 0.$$

$E^\perp \subset F$. TAKE $g \notin F$ THEN $\exists A \subset (0,1)$ SUCH THAT $|A| > 0$ AND $\int_{\{|x|=r\}} g(x)dx \geq \delta > 0$ $\forall r \in A$. TO SHOW $g \notin E^\perp$ WE NEED SOME $f \in E$ SUCH THAT $\int_{B_1} fg \neq 0$. TAKE $f = \chi_A$ $\left(f(x) = \begin{cases} 1 & |x| \in A \\ 0 & |x| \notin A \end{cases} \right)$

$$\int fg = \int_{|x| \in A} g = \int_A \Big(\underbrace{\int_{|x|=r} g(x)dx}_{\geq \delta} \Big) dr \geq |A| \delta > 0 \Rightarrow g \notin E^\perp$$

2) SHOW THAT, FOR ANY $f \in H$, WE HAVE

$$A(f) = f(x) - \fint_{\{|y|=|x|\}} f(y)dy \in E^\perp \qquad \left(\fint_A f := \frac{1}{\mu(A)} \int_A f \right)$$

AND DEDUCE AN EXPLICIT EXPRESSION FOR $P: H \to E$, $Q: H \to E^\perp$

LET US SHOW THAT $A(f) \in F$, THAT IS $\int_{|x|=r} A(f) = 0$ FOR A.E. r

$$\int_{|x|=r} A(f) = \int_{|x|=r} \Big(f(x) - \fint_{|y|=|x|} f(y)dy \Big) dx = \int_{|x|=r} f(x)dx - \cancel{|\{|x|=r\}|} \cancel{\fint}\!\!\int_{|y|=r} f(y)dy = 0$$

FOR ANY f, WE HAVE $f = \underset{\in E^\perp}{A(f)} + \underset{\in E}{\fint_{|y|=|x|} f(y)dy}$

SINCE $H = E^\perp \oplus E$, WE MUST HAVE $Q: f \to A(f)$

$P: f \to \fint f(y)dy$

REMARK

$$P: f \to \fint_{|y|=|x|} f(y)\,dy$$

REMARK

$N=1$ $(B_1(0)=(-1,1)) \Rightarrow E=\{\text{EVEN FUNCTIONS}\}$
$E^\perp=\{\text{ODD FUNCTIONS}\}$

EXERCISE 1 ON HAHN-BANACH'S SHEET

H HILBERT SPACE, $E \triangleleft H$, $L\in E^*$, $x_0\in E^\perp$
$L\not\equiv 0$ $\quad \|x_0\|=1$

$\tilde{L}\in \operatorname{span}\{E,x_0\}^*$ DEFINED BY $\tilde{L}(x+tx_0)=Lx+ct$.

1) FOR ANY $x\in E$ WITH $Lx\neq 0$, FIND THE MAXIMUM OF

$$t\to \frac{(\tilde{L}(x+tx_0))^2}{\|x+tx_0\|^2}=:f(t)$$

$$f(t)=\frac{(Lx+ct)^2}{\|x\|^2+t^2} \Rightarrow f'(t)=\frac{2(Lx+ct)(c\|x\|^2-tLx)}{(\|x\|^2+t^2)^2}$$

THE MAX IS ATTAINED AT $t_0=\frac{c\|x\|^2}{Lx}$, $f(t_0)=\frac{(Lx)^2}{\|x\|^2}+c^2$

2) SHOW THAT IF $c\neq 0$, THEN $\|\tilde{L}\|>\|L\|$, DEDUCE THAT L HAS A UNIQUE EXTENSION WHICH PRESERVES ITS NORM

$$\|\tilde{L}\|^2=\sup_{x,t}\frac{(\tilde{L}(x+tx_0))^2}{(\|x+tx_0\|)^2}\overset{1}{=}\sup_x\left(\frac{(Lx)^2}{\|x\|^2}+c^2\right)=\|L\|^2+c^2$$

$\Rightarrow \|\tilde{L}\|=\sqrt{\|L\|^2+c^2}$, IF $c\neq 0$ THEN $\|\tilde{L}\|>\|L\|$.

THEREFORE, ANY EXTENSION $\tilde{L}$ OF L WITH $\|\tilde{L}\|=\|L\|$ MUST SATISFY $\tilde{L}(x+tx_0)=Lx$ $\forall x_0\in E^\perp$ WITH $\|x_0\|=1$

THAT IS $\tilde{L}(x+y)=Lx$ $\forall y\in E^\perp$

THIS MEANS THAT THE ONLY EXTENSION WITH $\|\tilde{L}\|=\|L\|$ IS $\tilde{L}(z)=LPz$ ($P: H\to E$ IS THE ORTHOGONAL PROJ.)

EXERCISE 3 FROM SHEET 3

X NORMED SPACE, $E\subset X$ LINEAR SUBSPACE

DEFINE ... X^* ... E^*

X NORMED SPACE, $E \subset X$ LINEAR SUBSPACE

DEFINE $r: X^* \to E^*$
$$L \to L|_E$$

1) COMPUTE $\|r\|_{\mathcal{L}(X^*, E^*)} := \sup_{\|L\|_{X^*} \leq 1} \|r(L)\|_{E^*}$

IF $\|L\|_{X^*} \leq 1$, $|Lx| \leq 1$ FOR ANY $x \in X$ WITH $\|x\| \leq 1$

IN PARTICULAR, $|Lx| \leq 1$ $\forall x \in E$ WITH $\|x\| \leq 1$, THAT IS $\|r(L)\| \leq 1$

SO $\|r\|_{\mathcal{L}(X^*, E^*)} \leq 1$. LET US TRY AND PROVE $\|r\|_{\mathcal{L}(X,Y)} \geq 1$, THAT IS GIVEN $L_0 \in E^*$ WITH $\|L_0\|_{E^*} = 1$ WE LOOK FOR $\tilde{L} \in X^*$ SUCH THAT $L_0 = r(\tilde{L})$ AND $\|\tilde{L}\|_{X^*} = 1$. WE JUST EXTEND L_0 TO $\tilde{L} \in X^*$, $\tilde{L}|_{E^*} = L_0$, THAT IS $r(\tilde{L}) = L_0$ AND $\|\tilde{L}\| = \|L_0\| = 1$.

$\Rightarrow \|r\| \geq \dfrac{\|r(\tilde{L})\|}{\|\tilde{L}\|} = 1$, THEREFORE $\|r\| = 1$.

2) SHOW THAT r IS INJECTIVE $\iff$ E IS DENSE IN X

r IS INJECTIVE $\iff$ THE ONLY $L \in X^*$ SUCH THAT $r(L) = 0$ IS $L \equiv 0$, BUT $r(L) = L|_E$. WE KNOW THAT THE ONLY $L \in X^*$ SUCH THAT $L|_E \equiv 0$ IS $L \equiv 0$ IF ONLY IF E IS DENSE.

3) SHOW THAT r IS SURJECTIVE AND OPEN.

FOR ANY $L \in E^*$ WE LOOK FOR $M \in X^*$ SUCH THAT $r(M) = L$ SO WE JUST TAKE AS M THE HAHN-BANACH EXTENSION. $r(M) = M|_E$

WE SHOWED r IS CONTINUOUS AND SURJECTIVE. MOREOVER X^*, E^* ARE BOTH BANACH SPACES, AS THEY ARE DUALS, SO WE CAN APPLY THE OPEN MAP THEOREM TO GET r IS OPEN.

EXERCISE 3 FROM ESONERO

X NORMED SPACE AND $\{x_n\}$ SEQUENCE IN X SUCH THAT

$$\sum_{n \geq 1}^{+\infty} |Lx_n| < +\infty \qquad \forall L \in X^*$$

2) DEFINE $A: X^* \to$ THEN COMPUTE $\|A\|$.

$\sum_{k=1} |Lx_k| < +\infty \qquad \forall L \in X$

1) DEFINE $A_n : X^* \to \ell_1$ $\qquad \forall n \in \mathbb{N}$. COMPUTE $\|A_n L\|_{\ell_1}$

$\qquad L \to (Lx_1, \dots, Lx_n, 0 \dots 0)$ $\qquad$ AND SHOW A_n IS CONTINUOUS

$$\|A_n L\|_{\ell_1} := \sum_{k \in \mathbb{N}} |A L(k)| = \sum_{k=1}^{n} |Lx_k|.$$

SO $\|A_n L\| \leq \sum_{k=1}^{n} \|L\| \|x_k\| = \|L\| \underbrace{\left(\sum_{k=1}^{n} \|x_k\|\right)}_{= C_n} \Rightarrow \|A_n\|_{\mathcal{L}(X^*, \ell_1)} \leq C_n$

2) VERIFY THAT $\{A_n\}$ SATISFIES THE HYPOTHESES OF BANACH-STEINHAUS THEOREM AND APPLY THE THEOREM TO $\{A_n\}$

LET US VERIFY THAT $\sup \|A_n L\| < +\infty \quad \forall L \in X^*$:

$$\sup_n \|A_n L\| = \sup_n \sum_{k=1}^{n} |Lx_k| = \sum_{k=1}^{+\infty} |Lx_k| < +\infty \text{ BY HYPOTHESIS}$$

SINCE X^* IS A BANACH SPACE, WE CAN APPLY BANACH-STEINHAUS THEOREM TO GET $\sup_n \|A_n\|_{\mathcal{L}(X^*, \ell_1)} < +\infty$

3) SHOW THAT $\exists C > 0$ SUCH THAT $\sum_{k=1}^{+\infty} |Lx_k| < C\|L\| \quad \forall L \in X^*$

FROM 2) WE KNOW THAT $\underbrace{\|A_n L\|}_{= \sum_{k=1}^{n} |Lx_k|} \leq C\|L\|$

LET $n \to \infty$ AND WE GET

$$\sum_{k=1}^{\infty} |Lx_k| \leq C \|L\|$$

4) SHOW THAT, UNDER THE SAME HYPOTHESES, WE MAY NOT HAVE $\sum_{n=1}^{\infty} \|x_n\| < +\infty$

(SUGG. USE $x_n = \frac{e_n}{n} \in \ell_2$)

LET US VERIFY THE HYPOTHESES ARE SATISFIED.

ANY $L \in (\ell_2)^*$ IS $L : x \to \sum_{k=1}^{+\infty} x(k) y(k)$, SO

$$\sum_{k=1}^{+\infty} |Lx_n| = \sum_{n=1}^{+\infty} \left| \sum_{k=1}^{\infty} x_n(k)\, y(k) \right| = \sum_{n=1}^{+\infty} \frac{|y(n)|}{n} \underset{\text{CAUCHY-SCHWARTZ}}{\leq} \sqrt{\sum_{k=1}^{+\infty} |y(k)|^2} \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} < +\infty$$

BUT $\|x_n\| = \frac{1}{n}$, SO $\sum_{n=1}^{+\infty} \|x_n\| = +\infty$.