ZORN'S LEMMA LET $(P, \preceq)$ A PARTIAL ORDERED ($P \neq \emptyset$) SET SUCH THAT ANY $Q \subset P$ TOTALLY ORDERED HAS AN UPPER BOUND m ($m \in P$: $q \preceq m \ \forall q \in Q$).
THEN P HAS A MAXIMAL M ($\nexists p \in P$: $M \preceq p$)

PROP ANY HILBERT SPACE HAS A COMPLETE ORTHONORMAL FRAME.

PROOF $P = \{$ORTHONORMAL FRAMES$\}$ $\preceq \, = \, \subseteq$ INCLUSION
$P \neq \emptyset$ BECAUSE $\{e\} \in P$ IF $\|e\| = 1$
TAKE A TOTALLY ORDERED Q, AN UPPER BOUND IS $E_0 := \bigcup_{E \in Q} E$. E_0 IS AN O.N. FRAME: IF $e \in E_0$, $e \in E$ FOR SOME $E \in Q$. $\Rightarrow \|e\| = 1$; IF $e, f \in E_0$, $e \in E$ $f \in F$ FOR $E, F \in Q$
SINCE Q IS TOTALLY ORDERED, $E \subset F$ $\Rightarrow e, f \in F$
$\Rightarrow e \perp f$. THEREFORE $E_0 \in P$. BY CONSTRUCTION, $E \preceq E_0$ $\forall E \in Q \Rightarrow E_0$ IS UPPER BOUND.
BY ZORN'S LEMMA, $\exists \tilde{E} \in P$ MAXIMAL. WE SHOW $\tilde{E}$ IS COMPLETE. IF NOT, $\mathrm{SPAN}\, \tilde{E} \neq H$, SO $\mathrm{SPAN}\, \tilde{E}^{\perp} \neq \{0\}$
$e \in \mathrm{SPAN}\, \tilde{E}^{\perp}$, $\|e\| = 1 \Rightarrow \tilde{E} \cup \{e\}$ IS AN ORTHONORMAL FRAME CONTAINING $\tilde{E}$, CONTRADICTING MAXIMALITY.

COROLLARY H HAS A COUNTABLE (COMPLETE) ORTHONORMAL FRAME IF AND ONLY IF H IS SEPARABLE (HAS A COUNTABLE DENSE SUBSET)

EXAMPLES $L^2(-\pi, \pi)$ IS SEPARABLE.
$L^2(\#)$ ($\#$ COUNTING MEASURE OF X) IS SEPARABLE IF AND ONLY IF X IS COUNTABLE (FOR INSTANCE, ℓ_2)

PROOF $\Rightarrow$ SUPPOSE $\{e_n\}_{n \in \mathbb{N}}$ COMPLETE O.N. FRAME

SPAN [illegible]

$n \in \mathbb{N}$

$\text{SPAN}_{\mathbb{Q}} \{e_n\} := \{c_1 e_1 + \dots + c_N e_N;\ c_i \in \mathbb{Q}\}$ IS COUNTABLE AND DENSE

($\Leftarrow$) SUPPOSE $\{e_\alpha\}_{\alpha \in A}$ MORE THAN COUNTABLE O.N. FRAME

$\|e_\alpha - e_\beta\| = \sqrt{2} \quad \forall \alpha \neq \beta \Rightarrow B_{\frac{\sqrt{2}}{2}}(e_\alpha) \cap B_{\frac{\sqrt{2}}{2}}(e_\beta) = \emptyset$

$\Rightarrow \{B_{\frac{\sqrt{2}}{2}}(e_\alpha)\}_{\alpha \in A}$ PAIRWISE DISJOINTED OPEN SETS

THEN, ANY DENSE $D \subset H$ MUST INTERSECT ALL $B_{\frac{\sqrt{2}}{2}}(e_\alpha) \Rightarrow$ IT CANNOT BE COUNTABLE.

THEOREM (CHARACTERIZATION OF COMPLETE O.N. FRAMES)

LET $\{e_\alpha\}$ BE AN O.N. FRAME ON H. THEN, THE FOLLOWING ARE EQUIVALENT:

(a) $\{e_\alpha\}$ IS COMPLETE

(b) $\{e_\alpha\}^\perp = \{0\}$

(c) $x = \sum_\alpha (x, e_\alpha) e_\alpha$

(d) $x = \sum_\alpha c_\alpha e_\alpha$ FOR SOME $c_\alpha \in \mathbb{R}$

(e) $\|x\|^2 = \sum_\alpha (x, e_\alpha)^2$ (BESSEL'S EQUALITY)

(f) $(x, y) = \sum_\alpha (x, e_\alpha)(y, e_\alpha)$ (PARSEVAL'S IDENTITY)

REMARK: IF $\{e_\alpha\}$ IS NOT COMPLETE, THE PREVIOUS FORMULAS BECOME:

$$Px = \sum_\alpha (x, e_\alpha) e_\alpha$$

$$\|Px\|^2 = \sum_\alpha (x, e_\alpha)^2$$

$$(Px, Py) = \sum_\alpha (x, e_\alpha)(y, e_\alpha)$$

IN FACT, TAKE $\{f_\beta\}$ COMPLETE O.N. FRAME ON $\{e_\alpha\}^\perp$
$\Rightarrow x = \underbrace{\sum (x, e_\alpha) e_\alpha}_{\in \text{SPAN}\{e_\alpha\}} + \underbrace{\sum (y, f_\beta) f_\beta}_{\in \perp}$ BECAUSE $\{e_\alpha\} \cup \{f_\beta\}$ IS A COMPLETE O.N. FRAME ON H $\Rightarrow \underline{Px} = \sum_\alpha (x, e_\alpha) e_\alpha$.

PROOF: (a) $\Leftrightarrow$ (b) WE KNOW THAT $\overline{\text{SPAN } E} = H$ IF AND ONLY IF $E^\perp = \{0\}$. IN PARTICULAR, TAKE $E = \{e_\alpha\}$.

(b) $\Rightarrow$ (c) DEFINE $y = \sum_\alpha (x, e_\alpha) e_\alpha$

FOR ANY $\alpha \in A$, $(y, e_\alpha) = \left(\sum_\beta (x, e_\beta) e_\beta, e_\alpha \right) = (x, e_\alpha)$
$\Rightarrow x - y \perp e_\alpha$ BUT $\{e_\alpha\}^\perp = \{0\}$, SO $x = y = \sum_\alpha (x, e_\alpha) e_\alpha$

(c) $\Rightarrow$ (b) $x = \sum (x, e_\alpha) e_\alpha \ \forall x$, IN PARTICULAR

$\sum_\alpha (\cdot, e_\alpha) e_\alpha$

(c) $\Rightarrow$ (b) $x = \sum (x, e_\alpha) e_\alpha \ \forall x$, IN PARTICULAR $x \in \{e_\alpha\}^\perp$

$\Rightarrow (x, e_\alpha) = 0 \ \forall \alpha \Rightarrow x = 0$, SO $\{e_\alpha\}^\perp = 0$

(c) $\Rightarrow$ (d) OBVIOUS

(d) $\Rightarrow$ (c) SUPPOSE $x = \sum_\alpha c_\alpha e_\alpha$, SO FOR ANY $\alpha \in A$

$$\underline{(x, e_\alpha)} = \left(\sum_\beta c_\beta e_\beta, e_\alpha\right) = \underline{c_\alpha} \quad \Rightarrow x = \sum_\alpha c_\alpha e_\alpha = \sum_\alpha (x, e_\alpha) e_\alpha$$

(c) $\Leftrightarrow$ (e) $$\|x - \sum_{i=1}^N (x, e_{\alpha_i}) e_{\alpha_i}\|^2 = \|x\|^2 - \sum_{i=1}^N (x, e_{\alpha_i})^2$$

(AS IN THE PROOF OF BESSEL'S INEQUALITY)

LET $N \to \infty$. LEFT $\xrightarrow[N\to\infty]{} 0$ IF AND ONLY IF (c)

RIGHT $\xrightarrow[N\to\infty]{} 0$ IF AND ONLY IF (e)

(f) $\Rightarrow$ (e) JUST TAKE $x = y$

(e) $\Rightarrow$ (f) APPLY BESSEL'S EQUALITY TO $x, y, x+y$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2(x,y) = \cancel{\sum_\alpha (x, e_\alpha)^2} + \cancel{\sum_\alpha (y, e_\alpha)^2} + \cancel{2}\,\underline{(x,y)}$$

$$\| \qquad\qquad$$

$$\sum_\alpha (x+y, e_\alpha)^2 = \cancel{\sum_\alpha (x, e_\alpha)^2} + \cancel{\sum_\alpha (y, e_\alpha)^2} + \cancel{2}\,\underline{\sum_\alpha (x, e_\alpha)(y, e_\beta)}$$

COROLLARY LET $\{e_\alpha\}_{\alpha \in A}$ BE A COMPLETE O.N. FRAME ON H

THEN, $\exists\, \phi : H \to L^2(\#)$ SURJECTIVE ISOMETRY

WHERE $L^2(\#) = \{ f : A \to \mathbb{R} ;\ f(\alpha) \neq 0$ FOR AT MOST COUNTABLE α

$$\sum_{\alpha \in A} f(\alpha)^2 < +\infty \}$$

IN PARTICULAR, IF H IS SEPARABLE, IT IS ISOMETRIC TO ℓ_2

DIM $\qquad H \longrightarrow L^2(\#)$

$\qquad x \longrightarrow \phi(x) : \alpha \to (x, e_\alpha)$

$$x \longrightarrow \phi(x): \alpha \to (x, e_\alpha)$$

ϕ IS CLEARLY LINEAR

$$\underline{\|\phi(x)\|^2_{L^2(\#)}} = \int_A |\phi(x)|^2 \, d\# = \sum_\alpha (x, e_\alpha)^2 \overset{\text{BESSEL}}{=} \underline{\|x\|^2}$$

$\Rightarrow \phi$ IS AN ISOMETRY ($\Rightarrow$ INJECTIVE, CONTINUOUS)

ϕ IS SURJECTIVE: GIVEN $f \in L^2(\#)$, $x := \sum_\alpha f(\alpha) e_\alpha$

$\Rightarrow (x, e_\alpha) = f(\alpha) \Rightarrow \phi(x) = f$.

PROBLEM: EXTENDING $\underbrace{\text{LINEAR}}_{\text{CONTINUOUS}}$ FUNCTIONAL $L: E \to \mathbb{R}$

FROM $E \subset X$ LINEAR SUBSPACE TO ALL X (POSSIBLY $\|L\|_{E^*} = \|L\|_{X^*}$)

SOLUTION: HAHN-BANACH THEOREM

EASY CASE: E DENSE IN X

$\boxed{\text{PROP}}$ LET X BE A NORMED SPACE, $E \subset X$ LINEAR DENSE AND $L \in E^*$. THEN $\exists!\ \tilde{L} \in X^*$ SUCH THAT $\tilde{L}|_E = L$ AND $\|\tilde{L}\|_{X^*} = \|L\|_{E^*}$

<u>PROOF</u> GIVEN $x \in X$, $\exists\ x_n \to x$, $x_n \in E$. DEFINE $\tilde{L}(x) = \lim_{n \to \infty} Lx_n$

- $\tilde{L}(x)$ ESISTS: $|Lx_n - Lx_m| \leq \|L\| \|x_n - x_m\| \xrightarrow[n,m \to \infty]{} 0 \Rightarrow \{Lx_n\}$ IS CAUCHY $\Rightarrow \{Lx_n\}$ CONVERGES
- $\tilde{L}(x)$ DOES NOT DEPEND ON $\{x_n\}$: IF $x_n \to x$, $y_n \to x$, THEN BY CONTINUITY, $\lim_{n \to \infty} Lx_n = \lim_{n \to \infty} Ly_n$
- $\tilde{L}$ IS LINEAR: $\tilde{L}(\alpha x + \beta y) = \lim_{n \to \infty} L(\alpha x_n + \beta y_n) = \lim (\alpha L x_n + \beta L y_n)$

  $x_n \to x$, $y_n \to y \Rightarrow \alpha x_n + \beta y_n \to \alpha x + \beta y$

  $= \alpha \lim_{n \to \infty} Lx_n + \beta \lim_{n \to \infty} Ly_n = \alpha \tilde{L}(x) + \beta \tilde{L}(y)$
- $\tilde{L}$ IS CONTINUOUS...

$\lim_{n\to\infty} \ldots \lim_{n\to\infty} L y_n \ldots$

- $\tilde{L}$ IS CONTINOUS: TAKE x WITH $\|x\|=1$, $x_n \to x$

$\Rightarrow y_n := \frac{x_n}{\|x_n\|} \to x \Rightarrow \|\tilde{L}x\| = \|\lim L y_n\| = \lim \|L y_n\| \overset{\|y_n\|=1}{\leq} \|L\|$

TAKE $\sup_{\|x\|\leq 1}$ $\Rightarrow \|\tilde{L}\| \leq \|L\| < +\infty$

HOMEWORK! $\tilde{L}$ EXTENDS L, $\|\tilde{L}\| = \|L\|$, $\tilde{L}$ IS UNIQUE.