ADJOINT OF AN OPERATOR

$A: X \to Y \Rightarrow A^*: Y^* \to X^* \quad (A^* L)x = L(Ax) \quad \forall x \in X \ \forall L \in Y^*$

$X = Y$ HILBERT SPACE $\Rightarrow (Ax, y) = (x, A^* y) \quad \forall x, y \in X$

GENERALIZATION OF THE CONJUGATE TRANSPOSE

$\boxed{\text{EXAMPLES}}$ ① IF X, Y ARE FINITE-DIMENSIONAL, $A \in \mathcal{L}(X, Y)$ IS REPRESENTED WITH A MATRIX W.R.T. ORTHONORMAL BASES $\Rightarrow A^* \in \mathcal{L}(Y^*, X^*)$ IS REPRESENTED W.R.T. THE DUAL BASES WITH THE CONJUGATE TRANSPOSE

$a^*_{ij} = \overline{a_{ji}} \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \rightsquigarrow \begin{pmatrix} \overline{a_{11}} & \overline{a_{12}} \\ \overline{a_{21}} & \overline{a_{22}} \end{pmatrix}$

② $A = \mathbb{I} \in \mathcal{L}(X) \Rightarrow A^* = \mathbb{I} \in \mathcal{L}(X^*)$

③ $A =$ RIGHT SHIFT $\Rightarrow A^* =$ LEFT SHIFT $(X = \ell_2)$ $\quad (Ax, y) = \sum_{k \geq 2} x(k-1) y(k)$

$(x(1), x(2) \dots) \to (0, x(1) \dots) \qquad (y(1) y(2) \dots) \to (y(2), y(3) \dots) \qquad \| \quad (x, A^* y)$

④ $A: L^p(M) \to L^p(M) \quad g \in L^\infty(M) \Rightarrow A^*: L^{p'}(M) \to L^{p'}(M)$

$f \to fg \qquad \frac{1}{p} + \frac{1}{p'} = 1 \qquad h \to h\overline{g}$

$(Af)h = \int fg\overline{h} = \int f \overline{\overline{g} h} = A^* h$

$\boxed{\text{LEMMA}}$ LET $A \in \mathcal{L}(H)$ BE AN OPERATOR ON A COMPLEX HILBERT SPACE AND $A^* \in \mathcal{L}(H)$ BE ITS ADJOINT. THEN:

① $\ker A^* = (\operatorname{ran} A)^\perp$

② $\overline{\operatorname{ran} A^*} = (\ker A)^\perp$

③ $\sigma(A^*) = \operatorname{conj}(\sigma(A)) = \{\overline{\lambda}; \lambda \in \sigma(A)\}$

④ $A \in \underline{K(H)} \Leftrightarrow A^* \in K(H)$. IF SO, THEN $\operatorname{ran}(I - A^*) = \ker(I - A)^\perp$

IN PARTICULAR, $\dim \frac{H}{\operatorname{ran}(I - A^*)} = \dim(\ker(I - A))$

IN PARTICULAR, $\dim \frac{H}{\operatorname{ran}(I-A^*)} = \dim(\ker(I-A))$

PROOF ① WE WRITE $(Ax,y) = (x, A^*y)$

$y \in \ker A^* \iff (x, A^*y) = 0 \ \forall x \in H \iff (Ax, y) = 0 \ \forall x \in H \iff y \in (\operatorname{ran} A)^\perp$

$\Rightarrow \ker(A^*) = (\operatorname{ran} A)^\perp$

② $\overline{\operatorname{ran} A^*} = (\operatorname{ran} A^*)^{\perp\perp} \overset{①}{=} (\ker A^{**})^\perp = (\ker A)^\perp$

③ $(A-\lambda I)^* = A^* - \bar{\lambda} I$. $\lambda \notin \sigma(A^*) \iff A^* - \lambda I$ INVERTIBLE $\iff$

$(A^* - \lambda I)^* = A - \bar{\lambda} I$ IS INVERTIBLE $\iff \bar{\lambda} \notin \sigma(A) \iff \sigma(A^*) = \operatorname{Conj}(\sigma(A))$

④ ASSUME A TO BE COMPACT. TAKE $\{x_n\}_n$ BOUNDED, UP TO SUBSEQ.

$x_n \rightharpoonup x \Rightarrow A^* x_n \rightharpoonup A^* x$ BECAUSE LIN. OPERATOR ARE WEAKLY CONTINUOUS

$\left.\begin{array}{l}\{A^* x_n\} \text{ IS BOUNDED} \\ A \text{ IS COMPACT}\end{array}\right\} \Rightarrow AA^* x_n$ CONVERGES UP TO SUBSEQ.

IT MUST BE $AA^* x_n \to AA^* x$

WE WANT $A^* x_n \to Ax$.

$\|A^* x_n - A^* x\|^2 = (A^*(x_n - x), A^*(x_n - x)) = (x_n - x, AA^*(x_n - x)) \leq \underbrace{\|x_n - x\|}_{\leq C} \underbrace{\|AA^* x_n - AA^* x\|}_{\to 0}$

SINCE A^* IS COMPACT, THEN $\operatorname{ran}(I - A^*)$ IS CLOSED

$\operatorname{ran}(I - A^*) = \overline{\operatorname{ran}(I - A^*)} \overset{②}{=} (\ker(I - A^{**}))^\perp = \ker(I - A)^\perp$

SELF-ADJOINT OPERATORS

$\boxed{\text{DEF}}$ AN OPERATOR $A \in \mathcal{L}(X)$ ON A COMPLEX HILBERT SPACE IS SAID TO BE **SELF-ADJOINT** IF AND ONLY IF $A = A^*$, THAT IS $(Ax, y) = (x, Ay)$ $\forall x, y \in X$

$\boxed{\text{EXAMPLES}}$ ① IF $\dim X < +\infty$, THEN $A \in \mathcal{L}(X)$ IS SELF-ADJOINT IFF IT IS REPRESENTED BY A HERMITIAN MATRIX W.R.T. AN ORTHONORMAL BASE

$\bar{A}^t = A$

② $T \in \mathcal{L}(X)$ IS SELF-ADJ...

... W.R.T. AN ORTHONORMAL BASE $\quad \bar{A}^t = A$

② $I \in \mathcal{L}(X)$ IS SELF-ADJOINT

③ $A: L^2(M) \to L^2(M)$, $f \to fg$ IS SELF-ADJOINT IFF g IS REAL

④ $A: L^2([a,b]) \to L^2([a,b])$, $f \to u$, u SOLVES $\begin{cases} (-pu')' + qu = f \\ u(a) = u(b) = 0 \end{cases}$

IS SELF-ADJOINT BECAUSE $\int_a^b (Af)g = \int_a^b (Ag)f$.

PROPOSITION LET $A \in \mathcal{L}(H)$ BE A SELF-ADJOINT OPERATOR. THEN:

① $\sigma(A) \subset \mathbb{R}$

② $\sigma_2(A) = \emptyset$

③ λ, μ EIGENVALUES, x, y EIGENVECTORS $\begin{pmatrix} Ax = \lambda x & x \neq 0 \\ Ay = \mu y & y \neq 0 \end{pmatrix} \Rightarrow x \perp y$

④ $\sigma(A) \subset [m, M]$ $\quad m := \inf_{\|x\|=1} (Ax, x)$ $\quad M := \sup_{\|x\|=1} (Ax, x)$

⑤ $m, M \in \sigma(A) \Rightarrow \rho(A) = \max\{|m|, |M|\}$

⑥ $\|A\| = \rho(A)$

COROLLARY ① IF A IS SELF-ADJOINT AND $\sigma(A) = \{0\}$, THEN $A \equiv 0$.

② IF H IS SEPARABLE AND $A \in \mathcal{L}(H)$ IS SELF-ADJOINT, ITS EIGENVALUES ARE AT MOST COUNTABLE.

PROOF

① WE SAW $\sigma(A^*) = \mathrm{CONJ}(\sigma(A))$ BUT $A^* = A \Rightarrow \sigma(A) = \mathrm{Conj}(\sigma(A)) \Rightarrow \sigma(A) \subset \mathbb{R}$

② TAKE $\lambda \in \sigma(A)$. WE WANT TO SHOW THAT $\ker(A - \lambda I) = \{0\} \Rightarrow \mathrm{ran}(A - \lambda I)$ IS DENSE

$$\mathrm{ran}(A - \lambda I)^\perp = \ker((A - \lambda I)^*) \underset{\text{DEF BY ①}}{=} \ker(A - \lambda I) = \{0\} \Rightarrow \mathrm{ran}(A - \lambda I) \text{ IS DENSE.}$$

③ TAKE λ, μ SUCH THAT $Ax = \lambda x$, $Ay = \mu y$, $x, y \neq 0$

$$\Rightarrow (Ax, y) = (x, Ay) = (x, \mu y) = \bar{\mu}(x, y) \underset{(\mu \in \mathbb{R})}{=} \mu(x, y)$$

$$(Ax, y) = (\lambda x, y) = \lambda(x, y)$$

SINCE $\lambda \neq \mu \Rightarrow (x, y) = 0$, THAT IS $x \perp y$.

$\overline{\lambda(x,y)} = (x,y)$ $\qquad\qquad\qquad\qquad$ $\underline{M(x,y)}$

SINCE $\lambda \neq M \Rightarrow (x,y) = 0$, THAT IS $x \perp y$.

④ LET US SHOW FIRST THAT m, M ARE WELL-DEFINED, THAT IS $(Ax,x) \in \mathbb{R}\ \forall x$:

$$(Ax,x) = \overline{(x,Ax)} \underset{\substack{\uparrow \\ A \text{ IS SELF-ADJ.}}}{=} \overline{(Ax,x)} \Rightarrow (Ax,x) \text{ IS REAL. WE SUFFICE TO SHOW:}$$

$\lambda > M \Rightarrow \lambda \notin \sigma(A) \longrightarrow$ LET US PROVE THAT $A - \lambda I$ IS INJECTIVE.

($\lambda < m \Rightarrow \lambda \notin \sigma(A)$ CAN BE DONE AS FOR $\lambda > M$)

$$(A-\lambda I x, x) = (Ax,x) - \lambda \|x\|^2 \leq \underbrace{(M-\lambda)}_{<0}\|x\|^2 \Rightarrow \|(A-\lambda I)x\| \overset{\circledast}{\geq} (\lambda - M)\|x\| > 0$$

$\geq -\|(A-\lambda I)x\|\,\|x\|$ $\qquad (u,v) \geq -\|u\|\,\|v\|$

IF $x \neq 0$

$\Rightarrow (A-\lambda I)x \neq 0 \Rightarrow A-\lambda I$ IS INJECTIVE

WE ONLY NEED TO SHOW $\mathrm{ran}(A-\lambda I)$ IS CLOSED: $\left(\overset{②}{\Rightarrow} \mathrm{ran}(A-\lambda I) = \overline{\mathrm{ran}(A-\lambda I)} = X \Rightarrow A-\lambda I \text{ IS INJECTIVE AND SURJECTIVE} \right)$

TAKE $\{x_n\}$ SUCH THAT $\underline{(A-\lambda I)x_n \to y}$, LET US SHOW $y \in \mathrm{ran}(A-\lambda I)$:

BY $\circledast$ WE GET $\|x_n - x_m\| \leq \dfrac{\|(A-\lambda I)(x_n - x_m)\|}{\lambda - M} \xrightarrow[n,m\to\infty]{} 0$ BECAUSE $\{(A-\lambda I)x_n\}$ IS CAUCHY.

$x_n \to x_0$, BY CONTINUITY $\underline{(A-\lambda I)x_n \longrightarrow (A-\lambda I)x_0}$

$\Rightarrow y = (A-\lambda I)x_0$

EXAMPLE $\quad L^2([0,1]) \xrightarrow{A} L^2([0,1])$ $\quad (A: f \to fg,\ \underset{\substack{\downarrow\\ \text{REAL}}}{g(x) = x})$ IS SELF-ADJOINT

$f(x) \longrightarrow x f(x)$

$\sigma(A) \subset [m, M]$ $\qquad m = \inf_{\int |f|^2 = 1} \int x f(x)\overline{f(x)}\,dx$, $\quad M = \sup_{\int |f|^2 = 1} \int x f(x) \overline{f(x)}\,dx$

- $m = 0$: TAKE f_n CONCENTRATING AT $0/1$ $\quad f_n = \sqrt{n}\,\chi_{[0,\frac{1}{n}]}$ / $f_n = \sqrt{n}\,\chi_{[1-\frac{1}{n},1]}$
- $M = 1$

ACTUALLY, $\sigma(A) = [0,1]$. IN FACT, IF $\exists (A-\lambda I)^{-1} : L^2 \to L^2$,

THEN $\underset{\shortparallel}{(A-\lambda I)} f(x) = g(x) \Rightarrow g(x) = \dfrac{1}{x-\lambda} f(x) = (A-\lambda I)^{-1} g(x)$

THEN $\underset{\overset{\parallel}{(x-\lambda)f(x)}}{(A-\lambda I)f(x)} = g(x) \Rightarrow g(x) = \frac{1}{x-\lambda} f(x) = (A-\lambda I)^{-1} f(x)$

BUT $\frac{1}{x-\lambda} \notin L^2([0,1])$ IF $0 \leq \lambda \leq 1$, SO $\lambda \in \sigma(A)$

BUT $\nexists$ EIGENVALUES: $Af = \lambda f \Rightarrow x f(x) = \lambda f(x)$ A.E. $x \in [0,1]$

$\Rightarrow f \equiv 0.$