PROP LET $E \trianglelefteq X$ BE A CLOSED LINEAR SUBSPACE OF A BANACH SPACE. IF $\exists\, P \in \mathcal{L}(X,E)$ SUCH THAT $\underline{P^2 = P}$ THEN E HAS A COMPLEMENTARY SUBSPACE.

PROOF DEFINE $F := \operatorname{ran}(I_X - P) = \{x - Px \,;\, x \in X\}$

F IS A LINEAR SUBSPACE BECAUSE IT IS THE RANGE OF A LINEAR MAP. LET US SHOW $F = \overline{F}$: TAKE $y_n \in F$ SUCH THAT $y_n \to y_0$, $y_n = x_n - Px_n$. WE WANT $y_0 \in F$:

$$\begin{array}{c} y_n = (I-P)x_n \overset{\uparrow}{=} (I-P)(I-P)x_n = (I-P)y_n \longrightarrow \underline{(I-P)y_0} \\ \downarrow \qquad (I-P)^2 = I - 2P + P^2 = I - P \qquad\qquad \\ \underline{y_0} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \end{array}$$

IN PARTICULAR, $y_0 \in \operatorname{ran}(I-P) = F$

SO, F IS CLOSED

WE HAVE $X = E + F$ BECAUSE $\forall x \in X$ I HAVE $x = \underbrace{Px}_{\in E} + \underbrace{x - Px}_{\in F}$.

FINALLY, $E \cap F = \{0\}$. INFACT, ASSUME $z \in E \cap F$

$z = Px = y - \underline{Py}$ FOR SOME $x, y \in X$ $\Rightarrow P(x+y) = y \Rightarrow y \in \operatorname{ran} P$

$\Downarrow$

$Py = y$

$\Downarrow$

$z = y - Py = 0$.

PROP IF $E \trianglelefteq X$ HAS FINITE DIMENSION, THEN IT HAS A COMPLEMENTARY.

EXAMPLE THE SPACE F OF CONSTANT SEQUENCES HAS DIMENSION 1 AND IT HAS c_0 AS A COMPLEMENTARY IN c.

c_0 → INFINITESIMAL SEQUENCES

c → SEQUENCES HAVING FINITE LIMIT.

PROOF

LET $\{e_1, \dots, e_N\}$ BE A BASIS OF E, AND LET $\{L_1, \dots, L_N\} \subset E^*$ BE THE DUAL BASIS (SUCH THAT $L_i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$). WE EXTEND EACH L_i TO $\tilde{L}_i \in X^*$ USING HAHN-BANACH THEOREM AND WE DEFINE $P : X \to E$

$$x \longrightarrow (\tilde{L}_1 x)e_1 + \dots + (\tilde{L}_N x)e_N$$

$P \in \mathcal{L}(X,E)$ BECAUSE ALL $\tilde{L}_i$'S ARE LINEAR CONTINUOUS. IF $x \in E$, $x = c_1 e_1 + \dots + c_N e_N$

$Px = c_1 Pe_1 + \dots + c_N Pe_N = c_1 e_1 + \dots + c_N e_N = x \Rightarrow P$ IS A PROJECTION ON E, THEREFORE BY THE PREVIOUS PROPOSITION E HAS ...

$Px = c_1 e_1 + \dots c_N e_N = c_1 e_1 + \dots + c_N e_N = x \Rightarrow P$ IS A PROJECTION ON E, THEREFORE BY THE PREVIOUS PROPOSITION E HAS A COMPLEMENTARY.

THEOREM (CHARACTERIZATION OF INJECTIVE AND SURJECTIVE MAPS)

LET X, Y BE BANACH SPACE AND $A \in \mathcal{L}(X,Y)$.

1. IF $A \in \mathcal{L}(X,Y)$ IS SURJECTIVE, THEN IT HAS A CONTINUOUS RIGHT INVERSE $\iff \ker A$ HAS A COMPLEMENTARY IN X
2. IF $A \in \mathcal{L}(X,Y)$ IS INJECTIVE, THEN IT HAS A CONTINUOUS LEFT INVERSE $\iff \operatorname{ran} A$ IS CLOSED IN Y AND IT HAS A COMPLEMENTARY.

PROOF 1. ASSUME A HAS A CONTINUOUS RIGHT INVERSE $B \in \mathcal{L}(Y,X)$ SUCH THAT $A \circ B = I_Y$. LET US SHOW $\operatorname{ran}(B)$ IS A COMPLEMENTARY OF $\ker(A)$: $X = \ker A + \operatorname{ran} B$.

IN FACT, $x = \underbrace{x - BAx}_{\in \ker A} + \underbrace{BAx}_{\in \operatorname{ran} B}$ $\quad A(x - BAx) = Ax - (AB)Ax = Ax - Ax = 0$ $\Rightarrow x - BAx \in \ker A$

LET US SHOW $\ker A \cap \operatorname{ran} B = \{0\}$: TAKE $x \in \ker A \cap \operatorname{ran} B$ $\quad x = By,\ Ax = 0$

$\Rightarrow x = By \underset{AB = I}{=} BABy = BAx = B0 = 0. \Rightarrow \ker A \cap \operatorname{ran} B = \{0\} \Rightarrow \operatorname{ran} B$ IS A COMPLEMENTARY

NOW, ASSUME $\ker A$ HAS A COMPLEMENTARY F. BY THE PROPOSITION, $\exists P \in \mathcal{L}(X,F)$

WE SET $By = P(A^{-1}\{y\})$ $\quad A^{-1}\{y\}$ IS ANY x SUCH THAT $Ax = y$, IT IS WELL DEFINED BECAUSE IF $Ax' = y \Rightarrow x - x' \in \ker A \Rightarrow \underset{Px - Px'}{P(x - x')} = 0$

$\Rightarrow B$ DOES NOT DEPEND ON THE CHOICE.

LET US SHOW THAT $ABy = y \ \forall y \in Y$: SINCE A IS SURJECTIVE, $\exists x \in X$ SUCH THAT $Ax = y$ $\quad x = x_1 + x_2$ $\quad x_1 \in \ker A$, $x_2 = Px \in F$ $\Rightarrow ABy \underset{\text{def } P}{=} APx = Ax_2 \underset{x_1 \in \ker A}{=} Ax = y$

B IS CONTINUOUS BECAUSE OF THE CLOSED GRAPH THEOREM:

TAKE $y_n \to y$, $By_n \to x \in F$, I WANT $x = By$. APPLYING A, I GET $\underset{y_n \to y}{ABy_n} \to Ax$

$\Rightarrow By = BAx = PA^{-1}Ax = Px \underset{x \in F}{=} x.$

2. ASSUME A IS INJECTIVE AND $\exists B \in \mathcal{L}(Y,X)$ SUCH THAT $B \circ A = I_X$

$\operatorname{ran} A$ IS CLOSED: IF $Ax_n \to y$, $Ax_n = A\underbrace{BA}_{I}x_n \to ABy \in \operatorname{ran} A$

LET US SHOW $\ker B$ IS A COMPLEMENTARY OF ~~$\ker A$~~ $\operatorname{ran} A$

$Y = \operatorname{ran} A + \ker B$ BECAUSE $y = ABy + y - ABy$ $\cdot$ $ABy \in \operatorname{ran} A$

… OF … $\operatorname{ran} A$

$Y = \operatorname{ran} A + \ker B$ BECAUSE $y = ABy + y - ABy$; $ABy \in \operatorname{ran} A$

$y - ABy \in \ker B$ BECAUSE $B(y - ABy) = By - \underbrace{BA}_{I}By = By - By = 0$

FINALLY, $\ker B \cap \operatorname{ran} A = \{0\}$ BECAUSE $y \in \ker B \cap \operatorname{ran} A \Rightarrow y = Ax$, $By = 0$

$\Rightarrow y = Ax = ABAx = ABy = A0 = 0$

NOW, ASSUME $\operatorname{ran} A \triangleleft Y$ AND $Y = \operatorname{ran} A \oplus F$ FOR SOME $F \triangleleft Y$. I SET $B := A^{-1} P_Y$, WHERE $P \in \mathcal{L}(Y, \operatorname{ran} A)$ IS THE PROJECTION AND $A^{-1} : \operatorname{ran} A \to X$ IS WELL-DEFINED BECAUSE A IS INJECTIVE, MOREOVER A^{-1} IS CONTINUOUS BECAUSE $\operatorname{ran} A$ IS A BANACH SPACE, SINCE IT IS CLOSED IN Y. SO $B \in \mathcal{L}(Y, X)$

FINALLY, $\forall x \in X$ WE HAVE $BAx = A^{-1}PAx \underset{P|_{\operatorname{ran} A} = I}{=} A^{-1}Ax = x$.

EXAMPLE $\ell_2 \xrightarrow{A} \ell_2$ $\quad (Ax)(k) = \dfrac{x(k)}{k}$ $\quad A \in \mathcal{L}(\ell_2)$ BUT $\nexists B \in \mathcal{L}(\ell_2)$ SUCH THAT $BA = I_{\ell_2}$

$(x(1), x(2) \dots) \longrightarrow \left(x(1), \dfrac{x(2)}{2}, \dots \dfrac{x(k)}{k} \dots\right)$

IN FACT, $A(ke_k) = e_k \Rightarrow B(e_k) = ke_k \Rightarrow \|B\|_{\mathcal{L}(\ell_2)} \geq \|Be_k\| = k \quad \forall k \in \mathbb{N}$

$\Rightarrow \|B\| = +\infty$, THAT IS B IS NOT CONTINUOUS.

$\operatorname{ran} A$ IS NOT CLOSED: IN FACT, $e_k \in \operatorname{ran} A$, BY LINEARITY, $\underset{= c_{00} \text{ IS DENSE}}{\operatorname{span}\{e_k\}} \subset \operatorname{ran} A$

BUT $\operatorname{ran} A \neq \ell_2$ BECAUSE $y = \left(1, \dfrac{1}{2}, \dots \dfrac{1}{k} \dots\right) \notin \operatorname{ran} A$

IF $y \in \operatorname{ran} A$, $y = Ax \Rightarrow x(k) = 1 \quad \forall k \in \mathbb{N}$, BUT $x \notin \ell_2 \Rightarrow \operatorname{ran} A$ IS NOT CLOSED.

LEMMA IF $\{x_k\} \subset \ell_1$ VERIFIES $Lx_k \longrightarrow 0 \quad \forall L \in (\ell_1)^*$, THEN $\|x_k\|_{\ell_1} \xrightarrow[k \to \infty]{} 0$

$\left(\sum_{k \in \mathbb{N}} x_k(h)\, y(h) \longrightarrow 0 \quad \forall y \in \ell_\infty\right)$

EXAMPLE THERE EXISTS SURJECTIVE MAPS WITHOUT RIGHT INVERSE

$(\Rightarrow \exists$ SUBSPACES, SUCH AS $\ker A$, WITHOUT A COMPLEMENTARY$)$

$\ell_1 \xrightarrow{A} \ell_2$ $\quad$ TAKE $\{a_k\} \subset \ell_2$ DENSE IN THE UNIT BALL OF ℓ_2

$x \longrightarrow \displaystyle\sum_{k=1}^{\infty} x(k) a_k = \left(\sum_{k=1}^{\infty} a_k(1) x(k), \sum a_k(2) x(k) \dots\right)$

FIRST OF ALL, $Ax \in \ell_2$ BECAUSE $\|Ax\|_{\ell_2} = \left\|\sum a_k x(k)\right\| \leq \sum \|a_k\|_{\ell_2} |x(k)| = \|a_k\| \|x\|$

FIRST OF ALL, $Ax \in \ell_2$ BECAUSE $\|Ax\|_{\ell_2} = \|\sum_{k=1}^{\infty} a_k x(k)\| \leq \sum \|a_k\|_{\ell_2} |x(k)| = \|a_k\|_{\ell_2} \|x\|_{\ell_1}$

IN PARTICULAR, A IS CONTINUOUS (AND LINEAR)

LET US SHOW A IS SURJECTIVE: GIVEN $y \in \ell_2$, I LOOK FOR $x \in \ell_1$ SUCH THAT $Ax = y$. SINCE A IS LINEAR, I CAN ASSUME $\|y\| < 1$

SINCE $\{a_k\}$ IS DENSE, $\exists k_1 \in \mathbb{N}$ SUCH THAT $\|y - a_{k_1}\| < \frac{1}{2}$, THAT IS $y - a_{k_1} \in B_{\frac{1}{2}}$. SINCE $\{\frac{a_k}{2}\}$ IS DENSE IN $B_{\frac{1}{2}}$, $\|y - a_{k_1} - \frac{a_{k_2}}{2}\| < \frac{1}{4}$ FOR SOME $k_2 \in \mathbb{N}$

I REPEAT THE ARGUMENT $\Rightarrow \exists \{k_j\}$ SUCH THAT $\|y - a_{k_1} - \frac{a_{k_2}}{2} - \dots \frac{a_{k_j}}{2^j}\| < \frac{1}{2^{j+1}}$

LET $j \to +\infty$, $y = \sum_{j=1}^{+\infty} \frac{a_{k_j}}{2^j}$ BUT $a_k = A e_k \Rightarrow y = A\left(\sum_{j=1}^{+\infty} \frac{e_{k_j}}{2^j}\right) \in \operatorname{ran} A$.

LET US ASSUME BY CONTRADICTION $\exists B \in \mathcal{L}(\ell_2, \ell_1)$ SUCH THAT $AB = I_{\ell_2}$

TAKE $y \in \ell_\infty$ AND CONSIDER $x \to \sum_{k \in \mathbb{N}} (Bx)(k) y(k) \in (\ell_2)^* \Rightarrow \exists z_y \in \ell_2$ SUCH THAT $\sum (Bx)(k) y(k) = \sum x(k) z_y(k) \quad \forall x \in \ell_2$. TAKE $x = e_n$

$\sum (Be_n)(k) y(k) = \sum e_n(k) z(k) = z(n) \xrightarrow[n \to \infty]{} 0$.

BY THE LEMMA ($x_n = Be_n$) I HAVE $\|Be_n\|_{\ell_1} \xrightarrow[n \to \infty]{} 0$

$\underbrace{\|e_n\|}_{=1} = \|ABe_n\|_{\ell_2} \leq \|A\| \, \|Be_n\|_{\ell_1} \xrightarrow[n \to \infty]{} 0$, A CONTRADICTION.