SPECTRAL THEORY FOR COMPACT OPERATER

$$\begin{tikzcd} & A \\ \boxed{A-\lambda I} \arrow[ur, "\lambda=0"] \arrow[r, "\lambda\neq 0"'] & -\lambda\left(I-\frac{A}{\lambda}\right) \end{tikzcd}$$

$A \in K(X)$

SCALAR MULTIPLICATION = BIJECTION

$\boxed{I-A}$

$\boxed{\text{PROPOSITION}}$ LET X BE A COMPLEX BANACH SPACE AND $A \in K(X)$

THEN: ① $\dim(\ker(I-A)) < +\infty$

② $\operatorname{ran}(I-A)$ IS CLOSED

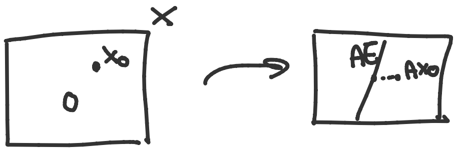
③ $\dim \dfrac{X}{\operatorname{ran}(I-A)} < +\infty$

$\boxed{\text{REMARK}}$ IF X IS A NORMED SPACE AND $E \trianglelefteq X$, THEN THE QUOTIENT $\frac{X}{E} := \{x+E;\, x \in X\}$ HAS A STRUCTURE OF NORMED SPACE $\|x+E\|_{\frac{X}{E}} := \inf\{\|x+y\|;\, y \in E\}$. IF X IS A BANACH SPACE, $\frac{X}{E}$ IS ALSO A BANACH SPACE.

↳ WE NEED E CLOSED

$A \in \mathcal{L}(X)$

$\boxed{\text{LEMMA}}$ LET X BE A BANACH SPACE AND $E \ntrianglelefteq X$ SUCH THAT $(I-A)X \subset E$. THEN $\exists\, x_0 \in X$ SUCH THAT $\|x_0\| = 1$, $\|\underline{Ax_0 - Ay}\| \geqslant \frac{1}{2}$ $\forall y \in E$



PROOF WE KNOW $\exists\, x_0 \in X$ SUCH THAT $\|\underline{x_0 - z}\| \geqslant \frac{1}{2}$ $\forall z \in E$ (OLD LEMMA OF QUASI-ORTHOGONALITY)

$$\forall y \in E,\ Ax_0 - Ay = x_0 - \underset{\in E}{\underbrace{y}} + \underbrace{(I-A)(x_0-y)}_{\in (I-A)X \subset E} \Rightarrow Ax_0 - Ay = x_0 - z \text{ FOR SOME } z \in E$$

$$\Downarrow$$

$$\|Ax_0 - Ay\| \geqslant \frac{1}{2}$$

PROOF OF THE PROP.

① ASSUME BY CONTRADICTION THAT $\ker(I-A)$ IS ∞-DIMENSIONAL. $\exists\, E_1 \subsetneq E_2 \subsetneq \dots E_n \subsetneq \dots$ INCREASING SEQUENCE OF CLOSED SUBSPACES. APPLYING THE LEMMA TO (E_n, E_{n+1}) WE FIND $x_{0,n} \in E_{n+1}$ SUCH THAT

APPLYING THE LEMMA TO (E_n, E_{n+1}) WE FIND $x_{0,n} \in E_{n+1}$ SUCH THAT $\|x_{0,n}\|=1$ AND $\|Ax_{0,n}-Ay\| \geq \frac{1}{2}$ $\forall y \in E_n$, IN PARTICULAR $\underline{\|Ax_{0,n}-Ax_{0,m}\| \geq \frac{1}{2}}$ $\forall m$

$\Rightarrow \{x_{0,n}\}$ IS BOUNDED BUT $\{Ax_{0,n}\}$ HAS NO CONVERGING SUBSEQ. BECAUSE OF CONTRADICTION WITH A BEING COMPACT.

② TAKE $\{x_n\}$ SUCH THAT $(I-A)x_n \to y$ FOR $y \in X$ AND WE WANT $y \in \mathrm{ran}(I-A)$. WE CLAIM $\underline{d_n := d(x_n, \mathrm{ran}(I-A))}$ IS BOUNDED.

IF NOT, $\exists\, z_n \in \ker(I-A)$ SUCH THAT $\|x_n - z_n\| \leq d_n + 1$, SO $\frac{x_n - z_n}{d_n}$ IS BOUNDED AND $A\left(\frac{x_n-z_n}{d_n}\right) \to w$ BECAUSE $A \in \mathcal{K}(X)$

$$d(w, \ker(I-A)) = \lim_{n\to\infty} d\Big(\frac{x_n - \overset{\in \ker(I-A)}{z_n}}{d_n}, \ker(I-A)\Big) = \lim_{n\to\infty} d\Big(\frac{x_n}{d_n}, \ker(I-A)\Big)$$

$$= \lim_{n\to\infty} \boxed{\frac{d(x_n, \ker(I-A))}{d_n}} = 1$$

$$w - Aw = \lim_{n\to\infty}\left(\frac{x_n - z_n}{d_n} - A\left(\frac{x_n-z_n}{d_n}\right)\right) = \lim_{n\to\infty}\left(\frac{\overbrace{x_n - Ax_n}^{\text{BOUNDED}}}{\underset{\to +\infty}{d_n}} - \frac{\overbrace{z_n - Az_n}^{=0}}{d_n}\right) = 0$$

IMPOSSIBLE BECAUSE $d(w, \ker(I-A)) = 1 \neq 0$.

SO $\exists\, z_n \in \ker(I-A)$ SUCH THAT $\|x_n - z_n\| \leq C \Rightarrow A(x_n - z_n) \to v$ BY COMPACTNESS

$$x_n - z_n = \underset{\downarrow\, y}{x_n - Ax_n} + \underset{\downarrow\, v}{A(x_n - z_n)} + \underbrace{(I-A)z_n}_{0}$$

$$\Rightarrow \underset{\underset{(I-A)x_n \,\to\, y}{\parallel}}{(I-A)(x_n - z_n)} \to (I-A)(y+v) \qquad \Rightarrow y \in (I-A)(y+v) \in \mathrm{ran}(I-A)$$

③ ASSUME BY CONTRADICTION $\dim \frac{X}{\mathrm{ran}(I-A)} = +\infty \Rightarrow \exists \{x_n + \mathrm{ran}(I-A)\}_{n\in\mathbb{N}}$ LIN. INDEPENDENT. IN $\frac{X}{\mathrm{ran}(I-A)}$. I APPLY THE LEMMA WITH

$E_n = \mathrm{SPAN}\{\underline{\mathrm{ran}(I-A)}, x_1, \dots, x_n\}$

$\underset{E_{n+1}}{X \supseteq} \mathrm{SPAN}\{\underline{\mathrm{ran}(I-A)}, x_1, \dots, x_n, x_{n+1}\}$

$\Rightarrow \exists\, x_{0,n} \in E_{n+1}$ WITH $\|x_{0,n}\|=1$ $\|Ax_{0,n} - Ax_{0,m}\| \geq \frac{1}{2}$ $\forall n, m$ CONTRADICTION WITH $A \in \mathcal{K}(X)$.

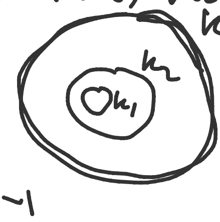
THEOREM (FREDHOLM ALTERNATIVE FOR COMPACT OPERATORS)

NOTATION WITH $A \in K(X)$.

THEOREM (FREDHOLM ALTERNATIVE FOR COMPACT OPERATORS)

LET X BE A BANACH SPACE AND $A \in K(X)$. DEFINE, $\forall n \in \mathbb{N}$

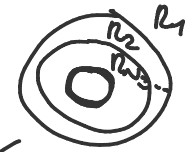
$K_n := \ker((I-A)^n)$ $\quad R_n = \operatorname{ran}((I-A)^n)$



$K_N = K_{N+1} = \dots$

① $\underline{\dim(K_n) < +\infty}$ AND $K_n \subset K_{n+1}$

$\underline{R_n \text{ IS CLOSED}}$, $\dim \frac{X}{R_n} < +\infty$ AND $R_n \subset R_{n-1}$



② $\exists N \in \mathbb{N}$ SUCH THAT $K_N = K_{N+1} = \dots$ ✓

$R_N = R_{N+1} = \dots$ ✓

$X = K_N \oplus R_N$ ✓

$(I-A) K_N \subset K_N$ NILPOTENT ✓

$(I-A) R_N \subset R_N$ INVERTIBLE ✓

③ $I-A$ INJECTIVE $\iff$ SURJECTIVE $(N=1, K_1 = \{0\}, R_1 = X)$

COROLLARY WE HAVE AN "ALTERNATIVE":

EITHER $I-A$ IS INJECTIVE AND $(I-A)x = y$ HAS A UNIQUE SOLUTION $\forall y \in X$

OR $(I-A)x = y$ HAS 0 SOL. IF $y \notin R_1$

HAS ∞ SOL. IF $y \in R_1$ OBTAINED BY ADDING $x \in \ker(I-A)$

(AFFINE SPACE OF DIM = DIM($\ker(I-A)$)).

EXAMPLE IN FINITE DIMENSION, THE THEOREM IS EQUIVALENT TO JORDAN'S CANONICAL FORM: ANY $A \in \mathcal{L}(\mathbb{C}^M)$ CAN BE WRITTEN, ON A SUITABLE BASIS,

$$A = \begin{pmatrix} A_1 & & 0 \\ & A_2 & \\ 0 & & \ddots \end{pmatrix} \qquad A_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}$$

IF $\lambda_j \neq 1$ $\forall j$, THEN $I-A$ IS INVERTIBLE

IF $\lambda_1 = 1 \Rightarrow K_1 = \{x_1 = 0\}$

$K_2 = \{x_1 = x_2 = 0\}$

$\vdots$

$K_{N_1} = \{x_1 = \dots = x_{N_1} = 0\} = K_n \quad n \geq N_1$

$R_{N_1} = \mathbb{C}^{M-N_1}$

$\dim(A_j) = N_j$ $\qquad$ λ_j NOT NECESSARILY DISTINCT

THE THEOREM APPLIES WITH $N = N_1$

PROOF OF FREDHOLM'S ALTERNATIVE

① $(I-A)^n = I - \underbrace{\sum_{k=1}^{n} (-1)^k \binom{n}{k} A^k}_{B_n}$

$(I-A)^2 = I - \underbrace{2A + A^2}_{\text{COMPACT}}$

B_n IS COMPACT BECAUSE COMBINATION OF COMPACT OPERATORS

$\Rightarrow$ BY THE PROPOSITION, $\dim(\ker(I-A)^n) < +\infty$

$\dim \frac{X}{\underbrace{\operatorname{ran}(I-A)^n}_{\text{CLOSED}}} < +\infty$

$K_N \subset K_{N+1}$ BECAUSE $x \in K_N \iff (I-A)^n x = 0 \Rightarrow (I-A)^{n+1}$

$R_N \subset R_{N-1}$ SIMILARLY

CLOSED

$K_N \subset K_{N+1}$ BECAUSE $x \in K_N \iff (I-A)^n x = 0 \Rightarrow (I-A)^{n+1} x = 0 \iff x \in K_{N+1}$

$R_N \subset R_{N-1}$ SIMILARLY

② $\overset{\text{STEP 1}}{\exists} N$ SUCH THAT $K_n = K_N \ \forall n \geq N$:

IF NOT, I APPLY THE LEMMA WITH $E = K_n$, $\cancel{X} = K_{n+1}$ $\forall n \in \mathbb{N}$

$\Rightarrow \exists x_n \in K_{n+1}$ SUCH THAT $\|x_n\| = 1$, $\|Ax_n - Ax_m\| \geq \frac{1}{2}$, IMPOSSIBLE BECAUSE $A \in K(X)$

STEP 2: $\exists M$ SUCH THAT $R_n = R_M \ \forall n \geq M$

AS BEFORE, IF NOT I APPLY THE LEMMA WITH $E = R_n$, ~~R~~ $X = R_{n-1}$, CONTRADICTION WITH $A \in K(X)$

STEP 3: $K_N \cap R_N = \{0\}$ IF N IS AS IN STEP 1

TAKE $y \in K_N \cap R_N$, THAT IS $(I-A)^N y = 0$, $y = (I-A)^N x \Rightarrow (I-A)^{2N} x = 0$

$\Rightarrow x \in K_{2N} = K_N \Rightarrow y = (I-A)^N x = 0 \Rightarrow K_N \cap R_N = \{0\}$

STEP 4: $M \leq N$ (M, N AS IN STEP 1, 2)

WE SUFFICE TO SHOW $K_{M-1} \cap R_{M-1} \neq \{0\}$, THANK TO STEP 4

TAKE $x \in R_{M-1} \setminus R_M \Rightarrow (I-A)x \in R_M = (I-A)R_M \Rightarrow \exists y \in R_M$ SUCH THAT

$(I-A)x = (I-A)y \Rightarrow x - y \neq 0$ BECAUSE $y \in R_M \not\ni x$

$\rightarrow x - y \in \ker(I-A) \subset K_{M-1}$

$x - y \in R_{M-1} + R_M = R_{M-1}$

$\Rightarrow K_{M-1} \cap R_{M-1} \neq \{0\}$

STEP 5: $X = K_N \oplus R_N$

WE KNOW $K_N \cap R_N = \{0\}$ (STEP 3), WE JUST NEED TO SHOW $X = K_N + R_N$

GIVEN $x \in X$, I FIND $y \in R_N$, $z \in K_N$ SUCH THAT $x = y + z$:

$(I-A)^N x \in R_N = R_{2N} = (I-A)^N R_N \Rightarrow \exists y \in R_N$ SUCH THAT $(I-A)^N x = (I-A)^N y$

$z := x - y \in K_N$ BECAUSE $(I-A)^N (x-y) = (I-A)^N x - (I-A)^N y = 0$

$\Rightarrow x = y + z \in R_N + K_N$

STEP 6: RESTRICTIONS ARE INVERTIBLE/NILPOTENT

$(I-A) K_N = K_{N-1} \subset K_N$, BY DEFINITON $(I-A)^N K_N \equiv 0$

$(I-A) R_N = R_{N+1} \subset R_N$, ACTUALLY $R_{N+1} = R_N$ SO $(I-A)|_{R_N}$ IS SURJECTIVE

LET US SHOW IT IS ALSO INJECTIVE: ASSUME $y \in R_N$, $(I-A)y = 0$

SINCE $R_N = R_{2N-1}$, $y = (I-A)^{N-1} x$, $x \in R_N \Rightarrow 0 = (I-A)y = (I-A)^N x$

$\Rightarrow x \in K_N \cap R_N = \{0\} \Rightarrow y = 0$

$\Rightarrow x \in K_N \cap W = \{0\} \Rightarrow y = 0$ $\qquad = (I-A)x$

③ ASSUME $I-A$ IS INJECTIVE $\Rightarrow K_1 = \{0\}, K_2 = \{0\}, \ldots \Rightarrow N = 1$, $X = \underset{\overset{\shortparallel}{\{0\}}}{K_1} \oplus R_1 \Rightarrow R_1 = X$

BUT $R_1 = \operatorname{ran}(I-A) \Rightarrow$ $I-A$ IS SURJECTIVE

ASSUME $I-A$ IS SURJECTIVE $\Rightarrow R_1 = X, R_2 = X, \ldots \Rightarrow X = K_N \oplus \underset{\overset{\shortparallel}{X}}{R_N}$ FOR SOME N

$\Rightarrow K_N = \{0\}$ BUT $K_1 \subset K_N = \{0\} \Rightarrow K_1 = \{0\} \Rightarrow$ $I-A$ IS INJECTIVE.