OPEN MAP THEOREM LET $A \in \mathcal{L}(X,Y)$ LINEAR CONTINUOUS BETWEEN X, Y BANACH SPACES. IF A IS SURJECTIVE, THEN A IS OPEN

COROLLARY ① IF X, Y ARE BANACH SPACES AND $A \in \mathcal{L}(X,Y)$ IS INVERTIBLE, THEN A^{-1} IS ~~INVERTIBLE~~ CONTINUOUS. $(\Rightarrow A^{-1} \in \mathcal{L}(Y,X))$

② IF $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ ARE BOTH BANACH SPACES AND $\exists C>0$ SUCH THAT $\|x\|_2 \leq C\|x\|_1$, $\forall x \in X$, THEN $\exists \tilde{C}>0$ SUCH THAT $\|x\|_1 \leq \tilde{C}\|x\|_2$

IN PARTICULAR, $\|\cdot\|_1$ AND $\|\cdot\|_2$ ARE EQUIVALENT.

PROOF ① FROM OPEN MAP THEOREM, $A(B_1(0)) \supset B_\delta(0) \Rightarrow B_1(0) \supset A^{-1}(B_\delta(0))$

THAT IS $\|y\| \leq \delta \Rightarrow \|A^{-1}y\| \leq 1$, DIVIDE BY δ $\underbrace{\|y\| \leq 1 \Rightarrow \|A^{-1}y\| \leq \frac{1}{\delta}}$

$$\|A^{-1}\| = \sup_{\|y\| \leq 1} \|A^{-1}y\| \leq \frac{1}{\delta} < +\infty$$

②
$$(X, \|\cdot\|_1) \xrightarrow{A} (X, \|\cdot\|_2)$$
$$x \longrightarrow x$$

BY HYPOTHESIS, $\underset{\|Ax\|}{\|x\|_2} \leq C\|x\|_1 = C\|x\| \Rightarrow A$ IS CONTINUOUS

SINCE $(X, \|\cdot\|_1)$ AND $(X, \|\cdot\|_2)$ ARE COMPLETE, I CAN APPLY ①

$$(X, \|\cdot\|_2) \xrightarrow{A^{-1}} (X, \|\cdot\|_1)$$
$$x \longrightarrow x$$

A^{-1} IS CONTINUOUS, THAT IS:

$$\underset{\|x\|_1}{\|A^{-1}x\|} \leq \tilde{C}\|x\| = \tilde{C}\|x\|_2$$

CLOSED GRAPH THEOREM LET X, Y BE BANACH SPACES AND $A: X \to Y$ LINEAR SUCH THAT ITS GRAPH

$$G_A := \{(x,y) \in X \times Y : y = Ax\}$$ IS CLOSED IN $X \times Y$.

THEN, A IS CONTINUOUS. ($X \times Y$ IS A BANACH SPACE WITH $\|(x,y)\|$

$\|x\|_X + \|y\|_Y$

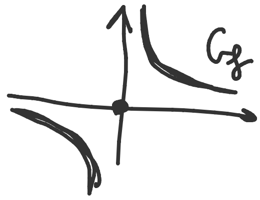
REMARK ① IN OTHER WORDS, IF $\begin{matrix} x_n \to x \\ Ax_n \to y \end{matrix} \Rightarrow Ax_n \to Ax$, THEN A IS CONTINUOUS

WITHOUT THE CLOSED GRAPH THEOREM, TO PROVE CONTINUITY I NEED

$x_n \to x \Rightarrow Ax_n \to Ax$ (FOR ANY $x_n \to x$, NOT ONLY THOSE SATISFYING $Ax_n \to y$.)

② THE THEOREM IS FALSE FOR NONLINEAR MAPS $f: \mathbb{R} \to \mathbb{R}$

f NOT CONTINUOUS BUT G_f IS CLOSED



$x \to \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

③ LINEAR MAPS MAY NOT BE CLOSED MAPS, THAT IS $A(Z)$ MAY NOT BE CLOSED IN Y EVEN IF $Z \subset X$ IS CLOSED.

FOR EXAMPLE, $X = C([0,1]) \to C([0,1]) = Y \Rightarrow$ NOW $A = \{f \in C([0,1]) : f(0) = 0\}$

$f \to \int_0^x f$ IS NOT CLOSED IN Y

PROOF G_A IS A LINEAR SUBSPACE OF $X \times Y$, IF IT IS CLOSED, IT IS A BANACH SPACE WITH $\|(x, Ax)\|_{G_A} := \|x\|_X + \|Ax\|_Y$. LET US CONSIDER

$\Pi: G_A \to X$ IT IS LINEAR, CONTINUOUS AND INVERTIBLE.

$(x, Ax) \to x$

BY THE COROLLARY, $\Pi^{-1}: X \to G_A$ IS CONTINUOUS, THAT IS

$x \longrightarrow (x, Ax)$

$$\|\Pi^{-1} x\| \leq C\|x\|$$
$$\|\Pi^{-1} x\| = \|x\| + \|Ax\|$$

$\Rightarrow \|Ax\| \leq (C-1)\|x\| \Rightarrow A$ IS CONTINUOUS.

COROLLARY LET X BE A BANACH SPACE AND $A: X \to X^*$ LINEAR SUCH THAT EITHER $(Ax)y = (Ay)x \quad \forall x, y \in X$ (SYMMETRIC)

OR $(Ax)x \geq 0 \quad \forall x \in X$ (NON-NEGATIVE)

THEN, A IS CONTINUOUS.

REMARK IN PARTICULAR, IF H IS A HILBERT SPACE, $A: H \to H$ LINEAR IS CONTINUOUS, PROVIDED $(Ax,y)=(Ay,x)$ OR $(Ax,x)\geq 0$ $\forall x,y$.

PROOF ① ASSUME A IS SYMMETRIC, WE WILL SHOW G_A IS CLOSED: TAKE x_n SUCH THAT $x_n \to x_0 \in X$, $Ax_n \to L_0 \in X^*$, WE WANT TO HAVE $L_0 = Ax_0$.

FOR ANY $x \in X$

$$\begin{array}{ccc} (Ax_n)x & \longrightarrow & L_0 x \\ \parallel & & \\ (Ax)x_n & \longrightarrow & (Ax)x_0 = (Ax_0)x \end{array}$$

SINCE $L_0 x = (Ax_0)x$ $\forall x \in X$, IT MUST BE $L_0 = Ax_0$

② ASSUME A NON-NEGATIVE AND TAKE $x_n \to x_0$ AND $Ax_n \to L_0$, APPLY A TO $x_n - x_0 - \lambda x$ FOR $\lambda \in \mathbb{R}$, $x \in X$:

$$0 \leq (A(x_n - x_0 - \lambda x))(x_n - x_0 - \lambda x) \longrightarrow (L_0 - Ax_0 - \lambda Ax)(-\lambda x) \underset{\parallel}{} $$

$$-\lambda (L_0 - Ax_0)x + \lambda^2 (Ax)x$$

$$\Rightarrow \lambda^2 (Ax)x \geq \lambda (L_0 - Ax_0)x$$

DIVIDE BY λ: $\lambda > 0 \Rightarrow (L_0 - Ax_0)x \leq \lambda (Ax)x \xrightarrow[\lambda \to 0]{} 0$

$\lambda < 0 \Rightarrow (L_0 - Ax_0)x \geq \lambda (Ax)x \xrightarrow[\lambda \to 0]{} 0$

$\Rightarrow (L_0 - Ax_0)x = 0$ $\forall x \in X \Rightarrow L_0 = Ax_0 \Rightarrow G_A$ IS CLOSED AND A IS CONTINUOUS

COMPLEMENTARY SUBSPACES AND PROJECTIONS

DEF LET X BE A BANACH SPACE AND $E \triangleleft X$. WE SAY THAT $F \triangleleft X$ IS A <u>COMPLEMENTARY</u> OF E IF $X = E + F$ AND $E \cap F = \{0\}$, AND WE WRITE $X = E \oplus F$

REMARKS ① BEING COMPLEMENTARY IS SYMMETRIC (IF F IS A COMPLEMENTARY OF E, THEN E IS A COMPLEMENTARY OF F)

② E, F ARE COMPLEMENTARY IFF I CAN WRITE ANY $x \in X$ <u>UNIQUELY</u> AS $x = y + z$ WITH $y \in E$, $z \in Z$

IF AND ONLY IF

EXAMPLES ① IF H IS A HILBERT SPACE ...

EXAMPLES ① IF H IS A HILBERT SPACE, ANY $E \lhd X$ HAS A COMPLEMENTARY GIVEN BY $F = E^{\perp}$

② $X \times Y \Rightarrow E = X \times \{0\}$ AND $F = \{0\} \times Y$ ARE COMPLEMENTARY

③ $X = C := \{x : \mathbb{N} \to \mathbb{R}$ SUCH THAT $\exists \lim_{k \to \infty} x(k) \in \mathbb{R}\}$ $\quad \|x\| = \sup_{k \in \mathbb{N}} |x(k)|$

$E = c_0 := \{x : \mathbb{N} \to \mathbb{R}$ SUCH THAT $\lim_{k \to \infty} x(k) = 0\}$ $\quad c_0 \lhd C$

$F := \{x : \mathbb{N} \to \mathbb{R}$ SUCH THAT $x(k) \equiv x_0 \ \forall k \in \mathbb{N}\}$ IS A COMPLEMENT OF c_0

CONSTANT SEQUENCES.

PROP "WHEN DOES A COMPLEMENTARY EXIST?"

LET X BE A BANACH SPACE AND $E \lhd X$. THEN, E HAS A COMPLEMENTARY $\iff \exists P \in \mathcal{L}(X, E)$ SUCH THAT $Px = x \ \forall x \in E$ (PROJECTION)

EXAMPLES ① IN HILBERT SPACES, P IS THE ORTHOGONAL PROJECTION

② IN $X \times Y$, $P : (x, y) \to (x, 0)$ ON $X \times \{0\}$

$(x, y) \to (0, y)$ ON $\{0\} \times Y$

③ $X = C, E = c_0 \Rightarrow P : X \to E$ $\quad P : X \to F$

$x(k) \to x(k) - \lim_{j \to \infty} x(j)$ $\quad x(k) \to \lim_{j \to \infty} x(j) \quad \forall k \in \mathbb{N}$

PROOF ASSUME E HAS A COMPLEMENTARY F, SO THAT $\forall x \in X \ \exists! \ x_E \in E$, $x_F \in F$ SUCH THAT $x = x_E + x_F$. DEFINE $Px = x_E$. $Px \in E$ BY CONSTRUCTION, $x \in E \Rightarrow x = \underset{\in E}{x} + \underset{\in F}{0} \Rightarrow Px = x$. LET US VERIFY P IS LINEAR!

$$\alpha x + \beta y = (\alpha x + \beta y)_E + (\alpha x + \beta y)_F \ *$$

$$\alpha(x_E + x_F) + \beta(y_E + y_F) = (\alpha x_E + \beta y_E) + (\alpha x_F + \beta y_F) \ *$$

$$\underbrace{(\alpha x + \beta y)_E - (\alpha x_E + \beta y_E)}_{\in E} = \alpha x_F + \beta y_F - (\alpha x + \beta y)_F \in F$$

$E \cap F = \{0\} \Rightarrow$ BOTH ARE ZERO $\Rightarrow (\alpha x + \beta y)_E = \alpha x_E + \beta y_E$

LET US SHOW P IS CONTINUOUS WITH CLOSED GRAPH THEOREM: $\Rightarrow Px = x_E$ IS LINEAR

$x_n \to x$, $Px_n \to y$, WE WANT $y = Px$.

$x - Px_n \to x - y$, F IS CLOSED

$x - Px_n \in F$ BECAUSE $Px_n \in E$

$\Rightarrow x - y \in F$.

NOW, WRITE $x = \underbrace{y}_{\in E} + \underbrace{x-y}_{\in F}$

$\quad\ \ = \underbrace{Px}_{\in E} + \underbrace{x - Px}_{\in F}$

SINCE $X = E \oplus F$, THE WRITING FORM IS UNIQUE, SO $y = Px$ $\Rightarrow$ CONTINUOUS

$x - y = x - Px$