On the normal form of the Kirchhoff equation

Pietro Baldi, Emanuele Haus

In memory of Walter Craig

Abstract. Consider the Kirchhoff equation

$$\partial_{tt} u - \Delta u \Big(1 + \int_{\mathbb{T}^d} |\nabla u|^2 \Big) = 0$$

on the d-dimensional torus $\mathbb{T}^d$. In a previous paper we proved that, after a first step of *quasi-linear* normal form, the resonant cubic terms show an integrable behavior, namely they give no contribution to the energy estimates. This leads to the question whether the same structure also emerges at the next steps of normal form. In this paper, we perform the second step and give a negative answer to the previous question: the quintic resonant terms give a nonzero contribution to the energy estimates. This is not only a formal calculation, as we prove that the normal form transformation is bounded between Sobolev spaces.

Keywords. Kirchhoff equation, quasilinear wave equations, quasilinear normal forms.
MSC2010: 35L72, 35Q74, 37J40, 70K45.

1 Introduction

We consider the Kirchhoff equation on the d-dimensional torus $\mathbb{T}^d$, $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (periodic boundary conditions)

$$\partial_{tt} u - \Delta u \Big(1 + \int_{\mathbb{T}^d} |\nabla u|^2 \, dx \Big) = 0. \tag{1.1}$$

Equation (1.1) is a quasilinear wave equation, and it has the structure of a Hamiltonian system

$$\begin{cases} \partial_t u = \nabla_v H(u,v) = v, \\ \partial_t v = -\nabla_u H(u,v) = \Delta u \Big(1 + \int_{\mathbb{T}^d} |\nabla u|^2 dx \Big), \end{cases} \tag{1.2}$$

where the Hamiltonian is

$$H(u,v) = \frac{1}{2} \int_{\mathbb{T}^d} v^2 dx + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2 dx + \Big(\frac{1}{2} \int_{\mathbb{T}^d} |\nabla u|^2 dx \Big)^2, \tag{1.3}$$

and $\nabla_u H$, $\nabla_v H$ are the gradients with respect to the real scalar product

$$\langle f, g \rangle := \int_{\mathbb{T}^d} f(x) g(x) \, dx \quad \forall f, g \in L^2(\mathbb{T}^d, \mathbb{R}), \tag{1.4}$$

namely $H'(u,v)[f,g] = \langle \nabla_u H(u,v), f \rangle + \langle \nabla_v H(u,v), g \rangle$ for all u, v, f, g. More compactly, (1.2) is

$$\partial_t w = J \nabla H(w), \tag{1.5}$$

where $w = (u, v)$, $\nabla H = (\nabla_u H, \nabla_v H)$ and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{1.6}$$

The Cauchy problem for the Kirchhoff equation is given by (1.1) with initial data at time $t = 0$

$$u(0, x) = \alpha(x), \quad u_t(0, x) = \beta(x). \tag{1.7}$$

Such a Cauchy problem is known to be locally well posed in time for initial data (α, β) in the Sobolev space $H^{\frac{3}{2}}(\mathbb{T}^d) \times H^{\frac{1}{2}}(\mathbb{T}^d)$ (see the work of Dickey [18]). However, the conserved Hamiltonian (1.3) only controls the $H^1 \times L^2$ norm of the couple (u, v). Since the local well-posedness has only been established in regularity higher than the energy space $H^1 \times L^2$, it is not trivial to determine whether the solutions are global in time. In fact, the question of global well-posedness for the Cauchy problem (1.1)-(1.7) with periodic boundary conditions (or with Dirichlet boundary conditions on bounded domains of $\mathbb{R}^d$) has given rise to a long-standing open problem: while it has been known for eighty years, since the pioneering work of Bernstein [7], that analytic initial data produce global-in-time solutions, it is still unknown whether the same is true for C^∞ initial data, even of small amplitude.

For initial data of amplitude ε, the linear theory immediately gives existence of the solution over a time interval of the order of ε^{-2}. In [4], we performed one step of quasilinear normal form and established a longer existence time, of the order of ε^{-4}; indeed, all the cubic terms giving a nontrivial contribution to the energy estimates are erased by the normal form. One may wonder whether the same type of mechanism works also for (one or more) subsequent steps of normal form.

In this paper, we give a negative answer to such a question, as we explicitly compute the second step of normal form for the Kirchhoff equation on $\mathbb{T}^d$, erasing all the nonresonant terms of degree five. It turns out that, differently from what happens for cubic terms, the contribution to the energy estimates of the resonant terms of degree five is different from zero. This, of course, leaves open the question whether for small amplitude initial data the time of existence can be extended beyond the lifespan $\sim \varepsilon^{-4}$ (partial results in this direction are in preparation [5]). The presence of resonant terms of degree five that give a nontrivial contribution to the energy estimates can, however, be interpreted as a sign of non-integrability of the equation. Another interesting open question is whether these "non-integrable" terms in the normal form can somehow be used to construct "weakly turbulent" solutions pushing energy from low to high Fourier modes, in the spirit of the works [11], [24], [25], [23], [22] for the semilinear Schrödinger equations on $\mathbb{T}^2$. Proving existence of such solutions may be a very hard task, but one may at least hope to use the normal form that we compute in this paper to detect some genuinely nonlinear behavior of the flow, over long time-scales (as in [20], [27]) or even for all times (as in [26]).

1.1 Main result

To give a precise statement of our main result, we introduce here the functional setting.

Function space. On the torus $\mathbb{T}^d$, it is not restrictive to set the problem in the space of functions with zero average in space, for the following reason. Given initial data $\alpha(x), \beta(x)$, we split both them and the unknown $u(t, x)$ into the sum of a zero-mean function and the average term,

$$\alpha(x) = \alpha_0 + \tilde{\alpha}(x), \quad \beta(x) = \beta_0 + \tilde{\beta}(x), \quad u(t, x) = u_0(t) + \tilde{u}(t, x),$$

where

$$\int_{\mathbb{T}^d} \tilde{\alpha}(x)\, dx = 0, \quad \int_{\mathbb{T}^d} \tilde{\beta}(x)\, dx = 0, \quad \int_{\mathbb{T}^d} \tilde{u}(t,x)\, dx = 0 \quad \forall t.$$

Then the Cauchy problem (1.1)-(1.7) splits into two distinct, uncoupled Cauchy problems: one is the problem for the average $u_0(t)$, which is

$$u_0''(t) = 0, \quad u_0(0) = \alpha_0, \quad u_0'(0) = \beta_0$$

and has the unique solution $u_0(t) = \alpha_0 + \beta_0 t$; the other one is the problem for the zero-mean component $\tilde{u}(t,x)$, which is

$$\tilde{u}_{tt} - \Delta \tilde{u} \Big(\int_{\mathbb{T}^d} |\nabla \tilde{u}|^2\, dx \Big) = 0, \quad \tilde{u}(0,x) = \tilde{\alpha}(x), \quad \tilde{u}_t(0,x) = \tilde{\beta}(x).$$

Thus one has to study the Cauchy problem for the zero-mean unknown $\tilde{u}(t,x)$ with zero-mean initial data $\tilde{\alpha}(x), \tilde{\beta}(x)$; this means to study (1.1)-(1.7) in the class of functions with zero average in x.

For any real $s \geq 0$, we consider the Sobolev space of zero-mean functions

$$H_0^s(\mathbb{T}^d, \mathbb{C}) := \Big\{ u(x) = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} u_j e^{ij \cdot x} : u_j \in \mathbb{C}, \ \|u\|_s < \infty \Big\}, \tag{1.8}$$

$$\|u\|_s^2 := \sum_{j \neq 0} |u_j|^2 |j|^{2s}, \tag{1.9}$$

and its subspace

$$H_0^s(\mathbb{T}^d, \mathbb{R}) := \{ u \in H_0^s(\mathbb{T}^d, \mathbb{C}) : u(x) \in \mathbb{R} \} \tag{1.10}$$

of real-valued functions u, for which the complex conjugates of the Fourier coefficients satisfy $\overline{u_j} = u_{-j}$. For $s = 0$, we write L_0^2 instead of H_0^0 the space of square-integrable functions with zero average.

Let $m_1 := 1$ if the dimension $d = 1$ and $m_1 := 2$ if $d \geq 2$. For $s \geq m_1$, $\delta > 0$, denote

$$B^s(\delta) := \big\{ (u,v) \in H_0^{s+\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \times H_0^{s-\frac{1}{2}}(\mathbb{T}^d, \mathbb{R}) \ : \ \max\{ \|u\|_{m_1 + \frac{1}{2}}, \|v\|_{m_1 - \frac{1}{2}} \} \leq \delta \big\},$$

$$B^s_{\mathrm{sym}}(\delta) := \{ (u,v) \in H_0^s(\mathbb{T}^d, \mathbb{C}) \times H_0^s(\mathbb{T}^d, \mathbb{C}) \ : \ v = \bar{u}, \ \|u\|_{m_1} \leq \delta \}.$$

In this paper we prove the following normal form result.

Theorem 1.1. *There exists $\delta > 0$ and a map $\Phi : B^{m_1}_{\mathrm{sym}}(\delta) \to B^{m_1}(2\delta)$, "close to identity" (see Remark 1.3), injective and conjugating system* (1.2) *to a system of the form*

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = W(u,v) = \mathcal{D}_1(u,v) + T_{\geq 3}(u,v) + W_5(u,v) + W_{\geq 7}(u,v). \tag{1.11}$$

The transformation Φ maps $B^s_{\mathrm{sym}}(\delta)$ to $B^s(2\delta)$ for all $s \geq m_1$. The vector field $\mathcal{D}_1$, defined in (4.2)*, is linear. The vector field $T_{\geq 3}$ contains only terms of homogeneity ≥ 3. Moreover, $\mathcal{D}_1$ and $T_{\geq 3}$ give no contribution to the energy estimates, namely the Sobolev norms of the solutions of the system $\partial_t(u,v) = \mathcal{D}_1(u,v) + T_{\geq 3}(u,v)$ are constant. The vector field W_5 contains only terms of homogeneity* 5*, it commutes with $\mathcal{D}_1$ and it gives a nonzero contribution to the energy estimates (see* (5.36)-(5.39)*). Finally, the vector field $W_{\geq 7}$ contains only terms of homogeneity ≥ 7.*

Remark 1.2. *Notation warning*: we are using the same notation (u, v) both for the original coordinates $(u, v) \in B^{m_1}(2\delta)$ in system (1.2) and for the final coordinates $(u, v) \in B^{m_1}_{\text{sym}}(\delta)$ in system (1.11), obtained after the normal form transformation Φ.

Remark 1.3. In Section 2 we will introduce the transformations $\Phi^{(1)}$ and $\Phi^{(2)}$, which symmetrize the system and introduce complex coordinates. These transformations are not close to identity. By saying that the map Φ is "close to identity" we mean that $\Phi = \Phi^{(1)} \circ \Phi^{(2)} \circ \Phi^{\text{next}}$, where Φ^{next} is bounded from $B^s_{\text{sym}}(\delta)$ to $B^s_{\text{sym}}(2\delta)$ for all $s \geq m_1$ and satisfies

$$\|(\Phi^{\text{next}} - \text{Id})(u, v)\|_s \leq C\|(u, v)\|^2_{m_1}\|(u, v)\|_s.$$

Remark 1.4. There is a certain similarity between our computation and the one performed by Craig and Worfolk [14] for the normal form of gravity water waves. In both cases one deals with an equation whose vector field is strongly unbounded (quasilinear here, fully nonlinear in [14]) and in both cases the first steps of normal form show an "integrable" behavior, while after few steps some genuinely non-integrable terms show up.

However, there is an important difference: while the normal form computed in [14] is only the result of a formal computation, the transformation Φ that we construct here to put the Kirchhoff equation in normal form is a bounded transformation that is well defined between Sobolev spaces. This is obtained thanks to the "quasilinear symmetrization" performed in [4], following the strategy for quasilinear normal forms introduced by Delort in the papers [16]-[17] on quasilinear Klein-Gordon equations on $\mathbb{T}$.

1.2 Related literature

Equation (1.1) was introduced by Kirchhoff [31] to model the transversal oscillations of a clamped string or plate, taking into account nonlinear elastic effects. The first results on the Cauchy problem (1.1)-(1.7) are due to Bernstein. In his 1940 pioneering paper [7], he studied the Cauchy problem on an interval, with Dirichlet boundary conditions, and proved global wellposedness for analytic initial data (α, β).

After that, the research on the Kirchhoff equation has been developed in various directions, with a different kind of results on compact domains (bounded subsets of $\mathbb{R}^d$ with Dirichlet boundary conditions, or periodic boundary conditions $\mathbb{T}^d$) or non compact domains ($\mathbb{R}^d$ or "exterior domains" $\Omega = \mathbb{R}^d \setminus K$, with $K \subset \mathbb{R}^d$ compact domain).

On $\mathbb{R}^d$, Greenberg and Hu [21] in dimension $d = 1$ and D'Ancona and Spagnolo [15] in higher dimension proved global wellposedness with scattering for small initial data in weighted Sobolev spaces.

On compact domains, dispersion, scattering and time-decay mechanisms are not available, and there are no results of global existence, nor of finite time blowup, for initial data (α, β) of Sobolev, or C^∞, or Gevrey regularity. The local wellposedness in the Sobolev class $H^{\frac{3}{2}} \times H^{\frac{1}{2}}$ has been proved by Dickey [18] (see also Arosio and Panizzi [2]), Beyond the question about the global wellposedness for small data in Sobolev class, another open question concerns the local wellposedness in the energy space $H^1 \times L^2$ or in $H^s \times H^{s-1}$ for $1 < s < \frac{3}{2}$.

We also mention the recent results [3], [34], [12], which prove the existence of time periodic or quasi-periodic solutions of time periodically or quasi-periodically forced Kirchhoff equations on $\mathbb{T}^d$, using Nash-Moser and KAM techniques.

For more details, generalizations and other open questions, we refer to Lions [32], to the surveys of Arosio [1], Spagnolo [35], Matsuyama and Ruzhansky [33], and to other references in our previous paper [4].

Concerning the normal form theory, and limiting ourselves to quasilinear PDEs on compact manifolds, we mention, in addition to the aforementioned papers of Delort [16]-[17], the abstract result of Bambusi [6] the recent literature on water waves by Craig and Sulem [13], Ifrim and Tataru [28], Ionescu and Pusateri [29]-[30], Berti and Delort [8], Berti, Feola and Pusateri [9]-[10], and the work by Feola and Iandoli [19] on the quasilinear NLS on $\mathbb{T}$.

Acknowledgements. We thank Roberto Feola for fruitful discussions on this subject. We also thank the anonymous referee for his/her useful comments. This research is supported by the INdAM-GNAMPA Project 2019.

2 Linear transformations

We start by recalling the first standard transformations in [4], which transform system (1.2) into another one (see (2.6)) where the linear part is diagonal, preserving both the real and the Hamiltonian structure of the problem. These standard transformations are the symmetrization of the highest order and then the diagonalization of the linear terms.

Symmetrization of the highest order. In the Sobolev spaces (1.8) of zero-mean functions, the Fourier multiplier

$$\Lambda := |D_x| : H_0^s \to H_0^{s-1}, \quad e^{ij\cdot x} \mapsto |j| e^{ij\cdot x}$$

is invertible. System (1.2) writes

$$\begin{cases} \partial_t u = v \\ \partial_t v = -(1 + \langle \Lambda u, \Lambda u \rangle)\Lambda^2 u, \end{cases} \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ is defined in (1.4); the Hamiltonian (1.3) is

$$H(u, v) = \frac{1}{2}\langle v, v\rangle + \frac{1}{2}\langle \Lambda u, \Lambda u\rangle + \frac{1}{4}\langle \Lambda u, \Lambda u\rangle^2.$$

To symmetrize the system at the highest order, we consider the linear, symplectic transformation

$$(u, v) = \Phi^{(1)}(q, p) = (\Lambda^{-\frac{1}{2}} q, \Lambda^{\frac{1}{2}} p). \tag{2.2}$$

System (2.1) becomes

$$\begin{cases} \partial_t q = \Lambda p \\ \partial_t p = -(1 + \langle \Lambda^{\frac{1}{2}} q, \Lambda^{\frac{1}{2}} q \rangle)\Lambda q, \end{cases} \tag{2.3}$$

which is the Hamiltonian system $\partial_t(q, p) = J\nabla H^{(1)}(q, p)$ with Hamiltonian $H^{(1)} = H \circ \Phi^{(1)}$, namely

$$H^{(1)}(q, p) = \frac{1}{2}\langle \Lambda^{\frac{1}{2}} p, \Lambda^{\frac{1}{2}} p\rangle + \frac{1}{2}\langle \Lambda^{\frac{1}{2}} q, \Lambda^{\frac{1}{2}} q\rangle + \frac{1}{4}\langle \Lambda^{\frac{1}{2}} q, \Lambda^{\frac{1}{2}} q\rangle^2, \quad J := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{2.4}$$

The original problem requires the "physical" variables (u, v) to be real-valued; this corresponds to (q, p) being real-valued, too. Also, note that $\langle \Lambda^{\frac{1}{2}} p, \Lambda^{\frac{1}{2}} p \rangle = \langle \Lambda p, p \rangle$.

Diagonalization of the highest order: complex variables. To diagonalize the linear part $\partial_t q = \Lambda p$, $\partial_t p = -\Lambda q$ of system (2.3), we introduce complex variables.

System (2.3) and the Hamiltonian $H^{(1)}(q, p)$ in (2.4) are also meaningful, without any change, for *complex* functions q, p. Thus we define the change of complex variables $(q, p) = \Phi^{(2)}(f, g)$ as

$$(q, p) = \Phi^{(2)}(f, g) = \Big(\frac{f + g}{\sqrt{2}}, \frac{f - g}{i\sqrt{2}} \Big), \qquad f = \frac{q + ip}{\sqrt{2}}, \quad g = \frac{q - ip}{\sqrt{2}}, \tag{2.5}$$

so that system (2.3) becomes

$$\begin{cases} \partial_t f = -i\Lambda f - i\frac{1}{4} \langle \Lambda(f + g), f + g \rangle \Lambda(f + g) \\ \partial_t g = i\Lambda g + i\frac{1}{4} \langle \Lambda(f + g), f + g \rangle \Lambda(f + g) \end{cases} \tag{2.6}$$

where the pairing $\langle \cdot, \cdot \rangle$ denotes the integral of the product of any two complex functions

$$\langle w, h \rangle := \int_{\mathbb{T}^d} w(x) h(x)\, dx = \sum_{j \in \mathbb{Z}^d \setminus \{0\}} w_j h_{-j}, \quad w, h \in L^2(\mathbb{T}^d, \mathbb{C}). \tag{2.7}$$

The map $\Phi^{(2)} : (f, g) \mapsto (q, p)$ in (2.5) is a $\mathbb{C}$-linear isomorphism of the space $L^2_0(\mathbb{T}^d, \mathbb{C}) \times L^2_0(\mathbb{T}^d, \mathbb{C})$ of pairs of complex functions. When (q, p) are real, (f, g) are complex conjugate. The restriction of $\Phi^{(2)}$ to the space

$$L^2_0(\mathbb{T}^d, c.c.) := \{(f, g) \in L^2_0(\mathbb{T}^d, \mathbb{C}) \times L^2_0(\mathbb{T}^d, \mathbb{C}) : g = \overline{f}\}$$

of pairs of complex conjugate functions is an $\mathbb{R}$-linear isomorphism onto the space $L^2_0(\mathbb{T}^d, \mathbb{R}) \times L^2_0(\mathbb{T}^d, \mathbb{R})$ of pairs of real functions. For $g = \overline{f}$, the second equation in (2.6) is redundant, being the complex conjugate of the first equation. In other words, system (2.6) has the following "real structure": it is of the form

$$\partial_t \begin{pmatrix} f \\ g \end{pmatrix} = \mathcal{F}(f, g) = \begin{pmatrix} \mathcal{F}_1(f, g) \\ \mathcal{F}_2(f, g) \end{pmatrix}$$

where the vector field $\mathcal{F}(f, g)$ satisfies

$$\mathcal{F}_2(f, \overline{f}) = \overline{\mathcal{F}_1(f, \overline{f})}. \tag{2.8}$$

Under the transformation $\Phi^{(2)}$, the Hamiltonian system (2.3) for complex variables (q, p) becomes (2.6), which is the Hamiltonian system $\partial_t(f, g) = iJ\nabla H^{(2)}(f, g)$ with Hamiltonian $H^{(2)} = H^{(1)} \circ \Phi^{(2)}$, namely

$$H^{(2)}(f, g) = \langle \Lambda f, g \rangle + \frac{1}{16} \langle \Lambda(f + g), f + g \rangle^2,$$

where J is defined in (2.4), $\langle \cdot, \cdot \rangle$ is defined in (2.7), and $\nabla H^{(2)}$ is the gradient with respect to $\langle \cdot, \cdot \rangle$. System (2.3) for real (q, p) (which corresponds to the original Kirchhoff equation) becomes system (2.6) restricted to the subspace $L^2_0(\mathbb{T}^d, c.c.)$ where $g = \overline{f}$.

To complete the definition of the function spaces, for any real $s \geq 0$ we define

$$H^s_0(\mathbb{T}^d, c.c.) := \{(f, g) \in L^2_0(\mathbb{T}^d, c.c.) : f, g \in H^s_0(\mathbb{T}^d, \mathbb{C})\}.$$

3 Diagonalization of the order one

In [4] (Section 3) the following global transformation $\Phi^{(3)}$ is constructed. Its effect is to remove the unbounded operator Λ from the "off-diagonal" terms of the equation, namely those terms coupling f and $\bar{f}$.

Lemma 3.1 (Lemma 3.1 of [4])**.** *Let $\Phi^{(3)}$ be the map*

$$\Phi^{(3)}(\eta,\psi) = \mathcal{N}(\eta,\psi)\begin{pmatrix}\eta\\ \psi\end{pmatrix}, \tag{3.1}$$

where $\mathcal{N}(\eta,\psi)$ is the matrix

$$\mathcal{N}(\eta,\psi) := \frac{1}{\sqrt{1-\rho^2(P(\eta,\psi))}}\begin{pmatrix}1 & \rho(P(\eta,\psi))\\ \rho(P(\eta,\psi)) & 1\end{pmatrix}, \tag{3.2}$$

ρ is the function

$$\rho(x) := \frac{-x}{1+x+\sqrt{1+2x}}, \tag{3.3}$$

P is the functional

$$P(\eta,\psi) := \varphi(Q(\eta,\psi)), \qquad Q(\eta,\psi) := \frac{1}{4}\langle\Lambda(\eta+\psi),\eta+\psi\rangle, \tag{3.4}$$

and φ is the inverse of the function $x\mapsto x\sqrt{1+2x}$, namely

$$x\sqrt{1+2x} = y \quad\Leftrightarrow\quad x = \varphi(y). \tag{3.5}$$

Then, for all real $s \geq \frac{1}{2}$, the nonlinear map $\Phi^{(3)} : H^s_0(\mathbb{T}^d, c.c.) \to H^s_0(\mathbb{T}^d, c.c.)$ is invertible, continuous, with continuous inverse

$$(\Phi^{(3)})^{-1}(f,g) = \frac{1}{\sqrt{1-\rho^2(Q(f,g))}}\begin{pmatrix}1 & -\rho(Q(f,g))\\ -\rho(Q(f,g)) & 1\end{pmatrix}\begin{pmatrix}f\\ g\end{pmatrix}.$$

For all $s \geq \frac{1}{2}$, all $(\eta,\psi) \in H^s_0(\mathbb{T}^d, c.c.)$, one has

$$\|\Phi^{(3)}(\eta,\psi)\|_s \leq C(\|\eta,\psi\|_{\frac{1}{2}})\|\eta,\psi\|_s$$

for some increasing function C. The same estimate is satisfied by $(\Phi^{(3)})^{-1}$.

In [4] it is proved that system (2.6), under the change of variable $(f,g) = \Phi^{(3)}(\eta,\psi)$, becomes

$$\begin{cases} \partial_t\eta = -i\sqrt{1+2P(\eta,\psi)}\,\Lambda\eta + \dfrac{i}{4(1+2P(\eta,\psi))}\Big(\langle\Lambda\psi,\Lambda\psi\rangle - \langle\Lambda\eta,\Lambda\eta\rangle\Big)\psi \\ \partial_t\psi = i\sqrt{1+2P(\eta,\psi)}\,\Lambda\psi + \dfrac{i}{4(1+2P(\eta,\psi))}\Big(\langle\Lambda\psi,\Lambda\psi\rangle - \langle\Lambda\eta,\Lambda\eta\rangle\Big)\eta. \end{cases} \tag{3.6}$$

Note that system (3.6) is diagonal at the order one, i.e. the coupling of η and ψ (except for the coefficients) is confined to terms of order zero. Also note that the coefficients of (3.6) are finite for $\eta,\psi \in H^1_0$, while the coefficients in (2.6) are finite for $f,g \in H^{\frac{1}{2}}_0$: the regularity threshold of the transformed system is $\frac{1}{2}$ higher than before. The real structure is preserved, namely the second equation in (3.6) is the complex conjugate of the first one, or, in other words, the vector field in (3.6) satisfies property (2.8).

Quintic terms. By Taylor's expansion,

$$\varphi(y) = y - y^2 + O(y^3) \quad (y \to 0). \tag{3.7}$$

Hence

$$\begin{aligned} P(\eta, \psi) &= Q(\eta, \psi) - Q^2(\eta, \psi) + O(Q^3(\eta, \psi)), \\ \frac{1}{1 + 2P(\eta, \psi)} &= 1 - 2Q(\eta, \psi) + 6Q^2(\eta, \psi) + O(Q^3(\eta, \psi)), \\ \sqrt{1 + 2P(\eta, \psi)} &= 1 + Q(\eta, \psi) - \frac{3}{2} Q^2(\eta, \psi) + O(Q^3(\eta, \psi)). \end{aligned} \tag{3.8}$$

The transformed Hamiltonian. Even if $\Phi^{(3)}$ is not symplectic, nonetheless it could be useful to calculate the transformed Hamiltonian, because it is still a prime integral of the equation. By definition (3.3), one has

$$\frac{\rho(x)}{1 - \rho^2(x)} = \frac{-x}{2\sqrt{1+2x}}, \quad \frac{1 + \rho^2(x)}{1 - \rho^2(x)} = \frac{1+x}{\sqrt{1+2x}} \quad \forall x \geq 0.$$

For $(f, g) = \Phi^{(3)}(\eta, \psi)$, one has

$$\langle \Lambda f, g \rangle = \frac{\rho(P(\eta, \psi))}{1 - \rho^2(P(\eta, \psi))} \Big(\langle \Lambda \eta, \eta \rangle + \langle \Lambda \psi, \psi \rangle \Big) + \frac{1 + \rho^2(P(\eta, \psi))}{1 - \rho^2(P(\eta, \psi))} \langle \Lambda \eta, \psi \rangle$$

and

$$\frac{1}{16} \langle \Lambda (f + g), f + g \rangle^2 = Q^2(f, g) = P^2(\eta, \psi).$$

Hence the new Hamiltonian $H^{(3)} := H^{(2)} \circ \Phi^{(3)}$ is

$$\begin{aligned} H^{(3)}(\eta, \psi) = & \frac{-P(\eta, \psi)}{2\sqrt{1 + 2P(\eta, \psi)}} \Big(\langle \Lambda \eta, \eta \rangle + \langle \Lambda \psi, \psi \rangle \Big) \\ & + \frac{1 + P(\eta, \psi)}{\sqrt{1 + 2P(\eta, \psi)}} \langle \Lambda \eta, \psi \rangle + P^2(\eta, \psi). \end{aligned}$$

4 Normal form: first step

The next step is the cancellation of the cubic terms contributing to the energy estimate. Following [4], we write (3.6) as

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = X(\eta, \psi) = \mathcal{D}_1(\eta, \psi) + \mathcal{D}_{\geq 3}(\eta, \psi) + \mathcal{B}_3(\eta, \psi) + \mathcal{R}_{\geq 5}(\eta, \psi) \tag{4.1}$$

where

$$\mathcal{D}_1(\eta, \psi) := \begin{pmatrix} -i\Lambda \eta \\ i \Lambda \psi \end{pmatrix}, \quad \mathcal{D}_{\geq 3}(\eta, \psi) := (\sqrt{1 + 2P(\eta, \psi)} - 1) \mathcal{D}_1(\eta, \psi), \tag{4.2}$$

$\mathcal{B}_3(\eta, \psi)$ is the cubic component of the bounded, off-diagonal term

$$\mathcal{B}_3(\eta, \psi) = \frac{i}{4} \Big(\langle \Lambda \psi, \Lambda \psi \rangle - \langle \Lambda \eta, \Lambda \eta \rangle \Big) \begin{pmatrix} \psi \\ \eta \end{pmatrix} \tag{4.3}$$

and $\mathcal{R}_{\geq 5}(\eta, \psi)$ is the bounded remainder of higher homogeneity degree

$$\mathcal{R}_{\geq 5}(\eta, \psi) = \frac{-iP(\eta, \psi)}{2(1 + 2P(\eta, \psi))} \Big(\langle \Lambda\psi, \Lambda\psi \rangle - \langle \Lambda\eta, \Lambda\eta \rangle \Big) \begin{pmatrix} \psi \\ \eta \end{pmatrix}. \tag{4.4}$$

In [4] the term $\mathcal{B}_3$ (and not $\mathcal{D}_{\geq 3}$, as it gives no contribution to the energy estimate) is removed by the following normal form transformation. Let

$$\Phi^{(4)}(w, z) := (I + M(w, z)) \begin{pmatrix} w \\ z \end{pmatrix}, \tag{4.5}$$

$$M(w, z) := \begin{pmatrix} 0 & A_{12}[w, w] + C_{12}[z, z] \\ A_{12}[z, z] + C_{12}[w, w] & 0 \end{pmatrix}, \tag{4.6}$$

where A_{12}, C_{12} are the bilinear maps

$$A_{12}[u, v]h := \sum_{j,k \neq 0,\, |j| \neq |k|} u_j v_{-j} \frac{|j|^2}{8(|j| - |k|)} h_k e^{ik\cdot x}, \tag{4.7}$$

$$C_{12}[u, v]h := \sum_{j,k \neq 0} u_j v_{-j} \frac{|j|^2}{8(|j| + |k|)} h_k e^{ik\cdot x}. \tag{4.8}$$

For $d \in \mathbb{N}$, let

$$m_0 = 1 \quad \text{if } d = 1, \qquad m_0 = \frac{3}{2} \quad \text{if } d \geq 2. \tag{4.9}$$

Lemma 4.1 (Lemma 4.1 of [4]). *Let A_{12}, C_{12}, m_0 be defined in (4.7), (4.8), (4.9). For all complex functions u, v, h, all real $s \geq 0$,*

$$\|A_{12}[u, v]h\|_s \leq \frac{3}{8} \|u\|_{m_0} \|v\|_{m_0} \|h\|_s, \quad \|C_{12}[u, v]h\|_s \leq \frac{1}{16} \|u\|_1 \|v\|_1 \|h\|_s. \tag{4.10}$$

The differential of $\Phi^{(4)}$ at the point (w, z) is

$$(\Phi^{(4)})'(w, z) = (I + K(w, z)), \qquad K(w, z) = M(w, z) + E(w, z), \tag{4.11}$$

where $M(w, z)$ is defined in (4.6), and

$$E(w, z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \begin{pmatrix} 2A_{12}[w, \alpha]z + 2C_{12}[z, \beta]z \\ 2C_{12}[w, \alpha]w + 2A_{12}[z, \beta]w \end{pmatrix}. \tag{4.12}$$

To estimate matrix operators and vectors in $H^s_0(\mathbb{T}^d, c.c.)$, we define $\|(w, z)\|_s := \|w\|_s = \|z\|_s$ for every pair $(w, z) = (w, \overline{w})$ of complex conjugate functions.

Lemma 4.2 (Lemma 4.2 of [4]). *For all $s \geq 0$, all $(w, z) \in H^{m_0}_0(\mathbb{T}^d, c.c.)$, $(\alpha, \beta) \in H^s_0(\mathbb{T}^d, c.c.)$ one has*

$$\Big\| M(w, z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Big\|_s \leq \frac{7}{16} \|w\|^2_{m_0} \|\alpha\|_s, \tag{4.13}$$

$$\Big\| K(w, z) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \Big\|_s \leq \frac{7}{16} \|w\|^2_{m_0} \|\alpha\|_s + \frac{7}{8} \|w\|_{m_0} \|w\|_s \|\alpha\|_{m_0}, \tag{4.14}$$

where m_0 is defined in (4.9)*. For $\|w\|_{m_0} < \frac{1}{2}$, the operator $(I + K(w,z)) : H_0^{m_0}(\mathbb{T}^d, c.c.) \to H_0^{m_0}(\mathbb{T}^d, c.c.)$ is invertible, with inverse*

$$(I + K(w,z))^{-1} = I - K(w,z) + \tilde{K}(w,z), \quad \tilde{K}(w,z) := \sum_{n=2}^{\infty} (-K(w,z))^n,$$

satisfying

$$\left\| (I + K(w,z))^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|_s \leq C(\|\alpha\|_s + \|w\|_{m_0} \|w\|_s \|\alpha\|_{m_0}),$$

for all $s \geq 0$, where C is a universal constant.

The nonlinear, continuous map $\Phi^{(4)}$ is invertible in a ball around the origin.

Lemma 4.3 (Lemma 4.3 of [4])**.** *For all $(\eta, \psi) \in H_0^{m_0}(\mathbb{T}^d, c.c.)$ in the ball $\|\eta\|_{m_0} \leq \frac{1}{4}$, there exists a unique $(w,z) \in H_0^{m_0}(\mathbb{T}^d, c.c.)$ such that $\Phi^{(4)}(w,z) = (\eta, \psi)$, with $\|w\|_{m_0} \leq 2\|\eta\|_{m_0}$. If, in addition, $\eta \in H_0^s$ for some $s > m_0$, then w also belongs to H_0^s, and $\|w\|_s \leq 2\|\eta\|_s$. This defines the continuous inverse map $(\Phi^{(4)})^{-1} : H_0^s(\mathbb{T}^d, c.c.) \cap \{\|\eta\|_{m_0} \leq \frac{1}{4}\} \to H_0^s(\mathbb{T}^d, c.c.)$.*

Lemma 4.4 (Lemma 4.4 of [4])**.** *For all complex functions u, v, y, h, one has*

$$\langle A_{12}[u,v]y, h \rangle = \langle y, A_{12}[u,v]h \rangle, \qquad \langle C_{12}[u,v]y, h \rangle = \langle y, C_{12}[u,v]h \rangle, \tag{4.15}$$

$$\overline{A_{12}[u,v]y} = A_{12}[\overline{u}, \overline{v}]\overline{y}, \qquad \overline{C_{12}[u,v]y} = C_{12}[\overline{u}, \overline{v}]\overline{y}, \tag{4.16}$$

$$[A_{12}[u,v], \Lambda^s] = 0, \qquad [C_{12}[u,v], \Lambda^s] = 0 \tag{4.17}$$

where $\overline{u}$ is the complex conjugate of u, and so on. Moreover, for all complex w, z,

$$M(w,z)\mathcal{D}_1 + \mathcal{D}_1 M(w,z) = 0. \tag{4.18}$$

Under the change of variables $(\eta, \psi) = \Phi^{(4)}(w,z)$, it is proved in [4] that system (3.6) becomes

$$\begin{aligned} \partial_t \begin{pmatrix} w \\ z \end{pmatrix} &= (I + K(w,z))^{-1} X(\Phi^{(4)}(w,z)) =: X^+(w,z) \\ &= \big(1 + \mathcal{P}(w,z)\big)\mathcal{D}_1(w,z) + X_3^+(w,z) + X_{\geq 5}^+(w,z) \end{aligned} \tag{4.19}$$

where

$$\mathcal{P}(w,z) := \sqrt{1 + 2P(\Phi^{(4)}(w,z))} - 1, \tag{4.20}$$

$X_3^+(w,z)$ has components

$$(X_3^+)_1(w,z) := -\frac{i}{4} \sum_{j,k \neq 0,\, |k|=|j|} w_j w_{-j} |j|^2 z_k e^{ik \cdot x}, \tag{4.21}$$

$$(X_3^+)_2(w,z) := \frac{i}{4} \sum_{j,k \neq 0,\, |k|=|j|} z_j z_{-j} |j|^2 w_k e^{ik \cdot x}, \tag{4.22}$$

and

$$\begin{aligned} X_{\geq 5}^+(w,z) := {} & K(w,z)\big(I + K(w,z)\big)^{-1}\big(\mathcal{B}_3(w,z) - X_3^+(w,z)\big) + \mathcal{R}_{\geq 5}^+(w,z) \\ & - \mathcal{P}(w,z)\big(I + K(w,z)\big)^{-1}\big(\mathcal{B}_3(w,z) - X_3^+(w,z)\big) \end{aligned} \tag{4.23}$$

with

$$\mathcal{R}^+_{\geq 5}(w,z) := (I+K(w,z))^{-1}\mathcal{R}_{\geq 5}(\Phi^{(4)}(w,z)) + [\mathcal{B}_3(\Phi^{(4)}(w,z)) - \mathcal{B}_3(w,z)] + \big(-K(w,z)+\tilde{K}(w,z)\big)\mathcal{B}_3(\Phi^{(4)}(w,z)), \tag{4.24}$$

$\mathcal{R}_{\geq 5}$ defined in (4.4).

Lemma 4.5 (Lemma 4.5 of [4])**.** *The maps $M(w,\overline{w})$, $K(w,\overline{w})$, and the transformation $\Phi^{(4)}$ preserve the structure of real vector field* (2.8)*. Hence X^+ defined in* (4.19) *satisfies* (2.8)*.*

The terms $(1+\mathcal{P})\mathcal{D}_1$ and X_3^+ in (4.19) give no contributions to the energy estimate, because, as one can check directly,

$$\langle \Lambda^s(1+\mathcal{P})(-i\Lambda w), \Lambda^s z\rangle + \langle \Lambda^s w, \Lambda^s(1+\mathcal{P})i\Lambda z\rangle = 0$$

and

$$\langle \Lambda^s(X_3^+)_1, \Lambda^s z\rangle + \langle \Lambda^s w, \Lambda^s(X_3^+)_2\rangle = 0. \tag{4.25}$$

Similarly, also $\mathcal{P}X_3^+$ gives no contribution to the energy estimate, because

$$\langle \Lambda^s(\mathcal{P}X_3^+)_1, \Lambda^s z\rangle + \langle \Lambda^s w, \Lambda^s(\mathcal{P}X_3^+)_2\rangle = \mathcal{P}\langle \Lambda^s(X_3^+)_1, \Lambda^s z\rangle + \mathcal{P}\langle \Lambda^s w, \Lambda^s(X_3^+)_2\rangle = 0.$$

Lemma 4.6 (Lemma 4.6 of [4])**.** *For all $s \geq 0$, all pairs of complex conjugate functions (w,z), one has*

$$\|\mathcal{B}_3(w,z)\|_s \leq \frac{1}{2}\|w\|_1^2\|w\|_s, \quad \|X_3^+(w,z)\|_s \leq \frac{1}{4}\|w\|_1^2\|w\|_s, \tag{4.26}$$

and, for $\|w\|_{m_0} \leq \frac{1}{2}$, for all complex functions h,

$$\|\mathcal{P}(w,z)h\|_s = \mathcal{P}(w,z)\|h\|_s, \quad 0 \leq \mathcal{P}(w,z) \leq C\|w\|^2_{\frac{1}{2}}, \tag{4.27}$$

$$\|\mathcal{R}_{\geq 5}(w,z)\|_s \leq 2P(w,z)\|\mathcal{B}_3(w,z)\|_s \leq C\|w\|^2_{\frac{1}{2}}\|w\|_1^2\|w\|_s \tag{4.28}$$

where $\mathcal{R}_{\geq 5}$ is defined in (4.4) *and C is a universal constant.*

Lemma 4.7 (Lemma 4.7 of [4])**.** *For all $s \geq 0$, all $(w,z) \in H^s_0(\mathbb{T}^d, c.c.) \cap H^{m_0}_0(\mathbb{T}^d, c.c.)$ with $\|w\|_{m_0} \leq \frac{1}{2}$, one has*

$$\|X^+_{\geq 5}(w,z)\|_s \leq C\|w\|_1^2\|w\|_{m_0}^2\|w\|_s \tag{4.29}$$

where C is a universal constant.

Quintic terms. Now we extract the terms of quintic homogeneity order from $X^+_{\geq 5}(w,z)$. Using (4.23), (4.24), (3.8), (3.4), (4.5), we calculate

$$X^+_{\geq 5}(w,z) = \mathcal{P}(w,z)X_3^+(w,z) + X_5^+(w,z) + X^+_{\geq 7}(w,z) \tag{4.30}$$

where

$$X_5^+(w,z) := -K(w,z)X_3^+(w,z) - 3Q(w,z)\mathcal{B}_3(w,z) + \mathcal{B}_3'(w,z)M(w,z)\begin{pmatrix} w \\ z\end{pmatrix} \tag{4.31}$$

and $X^+_{\geq 7}(w,z)$ is defined in (4.30) by difference. As already observed, the term $\mathcal{P}(w,z)X_3^+(w,z)$ in (4.30) gives no contributions to the energy estimate. By (4.19), (4.30), the complete vector field is

$$X^+(w,z) = (1+\mathcal{P}(w,z))\big(\mathcal{D}_1(w,z) + X_3^+(w,z)\big) + X_5^+(w,z) + X^+_{\geq 7}(w,z). \tag{4.32}$$

Moreover, adapting the proof of Lemma 4.7, we obtain the following bounds.

Lemma 4.8. *For all $s \geq 0$, all $(w, z) \in H_0^s(\mathbb{T}^d, c.c.) \cap H_0^{m_0}(\mathbb{T}^d, c.c.)$ with $\|w\|_{m_0} \leq \frac{1}{2}$, one has*

$$\|X_5^+(w,z)\|_s \leq C\|w\|_{m_0}^4 \|w\|_s, \qquad \|X_{\geq 7}^+(w,z)\|_s \leq C\|w\|_{m_0}^6 \|w\|_s,$$

where C is a universal constant.

We analyze the terms in (4.31). By (4.11), (4.12), the first component of $K(w,z)X_3^+(w,z)$ is

$$\begin{aligned}(K(w,z)X_3^+(w,z))_1 = A_{12}[w,w](X_3^+)_2(w,z) + C_{12}[z,z](X_3^+)_2(w,z) \\ + 2A_{12}[w,(X_3^+)_1(w,z)]z + 2C_{12}[z,(X_3^+)_2(w,z)]z,\end{aligned}$$

and its second component is the conjugate of the first one. Recalling (4.3), the first component of the last term in (4.31) is

$$\Big(\mathcal{B}_3'(w,z)M(w,z)\tbinom{w}{z}\Big)_1 = \frac{i}{2}\Big(\langle \Lambda z, \Lambda\beta\rangle - \langle \Lambda w, \Lambda\alpha\rangle\Big)z + \frac{i}{4}\Big(\langle \Lambda z, \Lambda z\rangle - \langle \Lambda w, \Lambda w\rangle\Big)\beta$$

with

$$\alpha = A_{12}[w,w]z + C_{12}[z,z]z, \quad \beta = A_{12}[z,z]w + C_{12}[w,w]w,$$

namely

$$\begin{aligned}\Big(\mathcal{B}_3'(w,z)M(w,z)\tbinom{w}{z}\Big)_1 &= \frac{i}{2}\langle \Lambda z, A_{12}[z,z]\Lambda w\rangle z + \frac{i}{2}\langle \Lambda z, C_{12}[w,w]\Lambda w\rangle z \\ &\quad - \frac{i}{2}\langle \Lambda w, A_{12}[w,w]\Lambda z\rangle z - \frac{i}{2}\langle \Lambda w, C_{12}[z,z]\Lambda z\rangle z \\ &\quad + \frac{i}{4}\langle \Lambda z, \Lambda z\rangle A_{12}[z,z]w + \frac{i}{4}\langle \Lambda z, \Lambda z\rangle C_{12}[w,w]w \\ &\quad - \frac{i}{4}\langle \Lambda w, \Lambda w\rangle A_{12}[z,z]w - \frac{i}{4}\langle \Lambda w, \Lambda w\rangle C_{12}[w,w]w.\end{aligned}$$

In Fourier series, with all indices in $\mathbb{Z}^d \setminus \{0\}$, one has

$$A_{12}[w,w](X_3^+)_2(w,z) = \frac{i}{32}\sum_{\substack{j,k,\ell \\ |j|\neq|k|=|\ell|}} \frac{|j|^2|\ell|^2}{|j|-|k|} w_j w_{-j} z_\ell z_{-\ell} w_k e^{ik\cdot x},$$

$$C_{12}[z,z](X_3^+)_2(w,z) = \frac{i}{32}\sum_{\substack{j,k,\ell \\ |k|=|\ell|}} \frac{|j|^2|\ell|^2}{|j|+|k|} z_j z_{-j} z_\ell z_{-\ell} w_k e^{ik\cdot x},$$

$$A_{12}[w,(X_3^+)_1(w,z)]z = \frac{-i}{32}\sum_{\substack{j,k,\ell \\ |\ell|=|j|\neq|k|}} \frac{|j|^2|\ell|^2}{|j|-|k|} w_j z_{-j} w_\ell w_{-\ell} z_k e^{ik\cdot x},$$

$$C_{12}[z,(X_3^+)_2(w,z)]z = \frac{i}{32}\sum_{\substack{j,k,\ell \\ |j|=|\ell|}} \frac{|j|^2|\ell|^2}{|j|+|k|} z_j w_{-j} z_\ell z_{-\ell} z_k e^{ik\cdot x},$$

$$Q(w,z) = \frac{1}{4}\sum_j |j|(w_j w_{-j} + 2w_j z_{-j} + z_j z_{-j}),$$

$$(\mathcal{B}_3(w,z))_1 = \frac{i}{4}\sum_{j,k} |j|^2(z_j z_{-j} - w_j w_{-j}) z_k e^{ik\cdot x},$$

$$\big(Q(w,z)\mathcal{B}_3(w,z)\big)_1 = \frac{i}{16}\sum_{j,k,\ell} |\ell||j|^2(w_\ell w_{-\ell} + 2w_\ell z_{-\ell} + z_\ell z_{-\ell})(z_j z_{-j} - w_j w_{-j}) z_k e^{ik\cdot x},$$

$$\langle \Lambda z, A_{12}[z,z]\Lambda w\rangle z = \frac{1}{8}\sum_{\substack{j,k,\ell\\ |\ell|\neq|j|}} \frac{|j|^2|\ell|^2}{|\ell|-|j|} z_j w_{-j} z_\ell z_{-\ell} z_k e^{ik\cdot x},$$

$$\langle \Lambda z, C_{12}[w,w]\Lambda w\rangle z = \frac{1}{8}\sum_{j,k,\ell} \frac{|j|^2|\ell|^2}{|\ell|+|j|} z_j w_{-j} w_\ell w_{-\ell} z_k e^{ik\cdot x},$$

$$\langle \Lambda w, A_{12}[w,w]\Lambda z\rangle z = \frac{1}{8}\sum_{\substack{j,k,\ell\\ |\ell|\neq|j|}} \frac{|j|^2|\ell|^2}{|\ell|-|j|} w_j z_{-j} w_\ell w_{-\ell} z_k e^{ik\cdot x},$$

$$\langle \Lambda w, C_{12}[z,z]\Lambda z\rangle z = \frac{1}{8}\sum_{j,k,\ell} \frac{|j|^2|\ell|^2}{|\ell|+|j|} w_j z_{-j} z_\ell z_{-\ell} z_k e^{ik\cdot x},$$

$$\langle \Lambda z, \Lambda z\rangle A_{12}[z,z]w = \frac{1}{8}\sum_{\substack{j,k,\ell\\ |k|\neq|j|}} \frac{|j|^2|\ell|^2}{|j|-|k|} z_j z_{-j} z_\ell z_{-\ell} w_k e^{ik\cdot x},$$

$$\langle \Lambda z, \Lambda z\rangle C_{12}[w,w]w = \frac{1}{8}\sum_{j,k,\ell} \frac{|j|^2|\ell|^2}{|j|+|k|} w_j w_{-j} z_\ell z_{-\ell} w_k e^{ik\cdot x},$$

$$\langle \Lambda w, \Lambda w\rangle A_{12}[z,z]w = \frac{1}{8}\sum_{\substack{j,k,\ell\\ |k|\neq|j|}} \frac{|j|^2|\ell|^2}{|j|-|k|} z_j z_{-j} w_\ell w_{-\ell} w_k e^{ik\cdot x},$$

$$\langle \Lambda w, \Lambda w\rangle C_{12}[w,w]w = \frac{1}{8}\sum_{j,k,\ell} \frac{|j|^2|\ell|^2}{|j|+|k|} w_j w_{-j} w_\ell w_{-\ell} w_k e^{ik\cdot x}.$$

Thus the first component of the quintic term $X_5^+(w,z)$ is

$$\begin{aligned}
(X_5^+(w,z))_1 = {}& -A_{12}[w,w](X_3^+)_2(w,z) - C_{12}[z,z](X_3^+)_2(w,z) \\
& - 2A_{12}[w,(X_3^+)_1(w,z)]z - 2C_{12}[z,(X_3^+)_2(w,z)]z \\
& - 3\big(Q(w,z)\mathcal{B}_3(w,z)\big)_1 \\
& + \frac{i}{2}\langle \Lambda z, A_{12}[z,z]\Lambda w\rangle z + \frac{i}{2}\langle \Lambda z, C_{12}[w,w]\Lambda w\rangle z \\
& - \frac{i}{2}\langle \Lambda w, A_{12}[w,w]\Lambda z\rangle z - \frac{i}{2}\langle \Lambda w, C_{12}[z,z]\Lambda z\rangle z \\
& + \frac{i}{4}\langle \Lambda z, \Lambda z\rangle A_{12}[z,z]w + \frac{i}{4}\langle \Lambda z, \Lambda z\rangle C_{12}[w,w]w \\
& - \frac{i}{4}\langle \Lambda w, \Lambda w\rangle A_{12}[z,z]w - \frac{i}{4}\langle \Lambda w, \Lambda w\rangle C_{12}[w,w]w
\end{aligned}$$

and, in Fourier series,

$$
\begin{aligned}
(X_5^+(w,z))_1 = &-\frac{i}{32}\sum_{\substack{j,k,\ell\\|j|\neq|k|=|\ell|}}\frac{|j|^2|\ell|^2}{|j|-|k|}w_jw_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x}\\
&-\frac{i}{32}\sum_{\substack{j,k,\ell\\|k|=|\ell|}}\frac{|j|^2|\ell|^2}{|j|+|k|}z_jz_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x}+\frac{i}{16}\sum_{\substack{j,k,\ell\\|\ell|=|j|\neq|k|}}\frac{|j|^2|\ell|^2}{|j|-|k|}w_jz_{-j}w_\ell w_{-\ell}z_ke^{ik\cdot x}\\
&-\frac{i}{16}\sum_{\substack{j,k,\ell\\|j|=|\ell|}}\frac{|j|^2|\ell|^2}{|j|+|k|}z_jw_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x}\\
&-\frac{3i}{16}\sum_{j,k,\ell}|\ell||j|^2(w_\ell w_{-\ell}+2w_\ell z_{-\ell}+z_\ell z_{-\ell})(z_jz_{-j}-w_jw_{-j})z_ke^{ik\cdot x}\\
&+\frac{i}{16}\sum_{\substack{j,k,\ell\\|\ell|\neq|j|}}\frac{|j|^2|\ell|^2}{|\ell|-|j|}z_jw_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x}+\frac{i}{16}\sum_{j,k,\ell}\frac{|j|^2|\ell|^2}{|\ell|+|j|}z_jw_{-j}w_\ell w_{-\ell}z_ke^{ik\cdot x}\\
&-\frac{i}{16}\sum_{\substack{j,k,\ell\\|\ell|\neq|j|}}\frac{|j|^2|\ell|^2}{|\ell|-|j|}w_jz_{-j}w_\ell w_{-\ell}z_ke^{ik\cdot x}-\frac{i}{16}\sum_{j,k,\ell}\frac{|j|^2|\ell|^2}{|\ell|+|j|}w_jz_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x}\\
&+\frac{i}{32}\sum_{\substack{j,k,\ell\\|k|\neq|j|}}\frac{|j|^2|\ell|^2}{|j|-|k|}z_jz_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x}+\frac{i}{32}\sum_{j,k,\ell}\frac{|j|^2|\ell|^2}{|j|+|k|}w_jw_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x}\\
&-\frac{i}{32}\sum_{\substack{j,k,\ell\\|k|\neq|j|}}\frac{|j|^2|\ell|^2}{|j|-|k|}z_jz_{-j}w_\ell w_{-\ell}w_ke^{ik\cdot x}-\frac{i}{32}\sum_{j,k,\ell}\frac{|j|^2|\ell|^2}{|j|+|k|}w_jw_{-j}w_\ell w_{-\ell}w_ke^{ik\cdot x}.
\end{aligned}
$$

Notation. In the coefficients of the vector field X_5^+ there appear several denominators, which imply the corresponding restrictions on the indices j,k,ℓ to prevent the denominators from vanishing. From now on, we will stop indicating explicitly the restrictions on the indices in summations and adopt instead the convention $0/0=0$ in the coefficients. For instance, instead of

$$\sum_{\substack{j,k,\ell\\|k|\neq|j|}}\frac{|j|^2|\ell|^2}{|j|-|k|}z_jz_{-j}w_\ell w_{-\ell}w_ke^{ik\cdot x}$$

we will write

$$\sum_{j,k,\ell}\frac{|j|^2|\ell|^2(1-\delta_{|j|}^{|k|})}{|j|-|k|}z_jz_{-j}w_\ell w_{-\ell}w_ke^{ik\cdot x}.$$

In this example, when $|j|=|k|$ the denominator of the coefficient vanishes; the numerator also vanishes because of the factor $(1-\delta_{|j|}^{|k|})$; this has to be interpreted as $\frac{|j|^2|\ell|^2(1-\delta_{|j|}^{|k|})}{|j|-|k|}$ being zero when $|j|=|k|$.

We collect similar monomials, and we get that $(X_5^+(w,z))_1$ is the sum of the following eight terms:

$$Y_{11}^{(4)}[w,w,w,w]w := -\frac{i}{32}\sum_{j,\ell,k}\frac{|j|^2|\ell|^2}{|j|+|k|}w_jw_{-j}w_\ell w_{-\ell}w_ke^{ik\cdot x}, \tag{4.33}$$

$$\begin{aligned}Y_{11}^{(2)}[w,w,z,z]w := \frac{i}{32}\sum_{j,\ell,k}|j|^2|\ell|^2\Big(&\frac{-\delta_{|\ell|}^{|k|}(1-\delta_{|j|}^{|k|})}{|j|-|k|}+\frac{1}{|j|+|k|}\\ &-\frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)w_jw_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x},\end{aligned} \tag{4.34}$$

$$Y_{11}^{(0)}[z,z,z,z]w := \frac{i}{32}\sum_{j,\ell,k}|j|^2|\ell|^2\Big(\frac{-\delta_{|\ell|}^{|k|}}{|j|+|k|}+\frac{(1-\delta_{|j|}^{|k|})}{|j|-|k|}\Big)z_jz_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x}, \tag{4.35}$$

$$Y_{12}^{(4)}[w,w,w,w]z := \frac{3i}{16}\sum_{j,\ell,k}|j|^2|\ell|w_jw_{-j}w_\ell w_{-\ell}z_ke^{ik\cdot x}, \tag{4.36}$$

$$\begin{aligned}Y_{12}^{(3)}[w,w,w,z]z := \frac{i}{16}\sum_{j,\ell,k}|j|^2|\ell|\Big(&\frac{|\ell|\delta_{|\ell|}^{|j|}(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}+6\\ &+\frac{|\ell|}{|\ell|+|j|}+\frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big)w_jw_{-j}w_\ell z_{-\ell}z_ke^{ik\cdot x},\end{aligned} \tag{4.37}$$

$$Y_{12}^{(2)}[w,w,z,z]z := \frac{3i}{16}\sum_{j,\ell,k}|j||\ell|(|j|-|\ell|)w_jw_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x}, \tag{4.38}$$

$$\begin{aligned}Y_{12}^{(1)}[w,z,z,z]z := \frac{i}{16}\sum_{j,\ell,k}|j||\ell|^2\Big(&\frac{-|j|\delta_{|j|}^{|\ell|}}{|j|+|k|}-6+\frac{|j|(1-\delta_{|j|}^{|\ell|})}{|\ell|-|j|}\\ &-\frac{|j|}{|\ell|+|j|}\Big)w_jz_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x},\end{aligned} \tag{4.39}$$

$$Y_{12}^{(0)}[z,z,z,z]z := -\frac{3i}{16}\sum_{j,\ell,k}|j|^2|\ell|z_jz_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x}. \tag{4.40}$$

Symmetrizing in $j\leftrightarrow\ell$ when it is possible, we also have

$$Y_{11}^{(4)}[w,w,w,w]w := -\frac{i}{64}\sum_{j,\ell,k}\Big(\frac{|j|^2|\ell|^2}{|j|+|k|}+\frac{|j|^2|\ell|^2}{|\ell|+|k|}\Big)w_jw_{-j}w_\ell w_{-\ell}w_ke^{ik\cdot x}, \tag{4.41}$$

$$\begin{aligned}Y_{11}^{(0)}[z,z,z,z]w := \frac{i}{64}\sum_{j,\ell,k}|j|^2|\ell|^2\Big(&-\frac{\delta_{|\ell|}^{|k|}+\delta_{|j|}^{|k|}}{|j|+|\ell|}\\ &+\frac{(1-\delta_{|j|}^{|k|})}{|j|-|k|}+\frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)z_jz_{-j}z_\ell z_{-\ell}w_ke^{ik\cdot x},\end{aligned} \tag{4.42}$$

$$Y_{12}^{(4)}[w,w,w,w]z := \frac{3i}{32}\sum_{j,\ell,k}|j||\ell|(|j|+|\ell|)w_jw_{-j}w_\ell w_{-\ell}z_ke^{ik\cdot x}, \tag{4.43}$$

$$Y_{12}^{(0)}[z,z,z,z]z := -\frac{3i}{32}\sum_{j,\ell,k}|j||\ell|(|j|+|\ell|)z_jz_{-j}z_\ell z_{-\ell}z_ke^{ik\cdot x}. \tag{4.44}$$

5 Normal form: second step

We consider a transformation of the form

$$\begin{pmatrix} w \\ z \end{pmatrix} = (I + \mathcal{M}(u,v)) \begin{pmatrix} u \\ v \end{pmatrix} =: \Phi^{(5)}(u,v), \tag{5.1}$$

where $\mathcal{M}(u,v)$ is a matrix operator of homogeneity degree 4. In particular,

$$\mathcal{M}(u,v) = \mathcal{A}[u,u,u,u] + \mathcal{B}[u,u,u,v] + \mathcal{C}[u,u,v,v] + \mathcal{D}[u,v,v,v] + \mathcal{F}[v,v,v,v], \tag{5.2}$$

where $\mathcal{A}[u,u,u,u]$ is of the form

$$\mathcal{A}[u,u,u,u] = \begin{pmatrix} \mathcal{A}_{11}[u,u,u,u] & \mathcal{A}_{12}[u,u,u,u] \\ \mathcal{A}_{21}[u,u,u,u] & \mathcal{A}_{22}[u,u,u,u] \end{pmatrix}$$

and similarly for the other terms and for $\mathcal{M}(u,v)$. We assume the following symmetries on the multilinearity of the maps $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{F}$:

$$\begin{aligned}
\mathcal{A}[u^{(1)}, u^{(2)}, u^{(3)}, u^{(4)}] &= \mathcal{A}[u^{(2)}, u^{(1)}, u^{(3)}, u^{(4)}] = \mathcal{A}[u^{(1)}, u^{(2)}, u^{(4)}, u^{(3)}], \\
\mathcal{B}[u^{(1)}, u^{(2)}, u^{(3)}, v] &= \mathcal{B}[u^{(2)}, u^{(1)}, u^{(3)}, v], \\
\mathcal{C}[u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}] &= \mathcal{C}[u^{(2)}, u^{(1)}, v^{(1)}, v^{(2)}] = \mathcal{C}[u^{(1)}, u^{(2)}, v^{(2)}, v^{(1)}], \\
\mathcal{D}[u, v^{(1)}, v^{(2)}, v^{(3)}] &= \mathcal{D}[u, v^{(1)}, v^{(3)}, v^{(2)}], \\
\mathcal{F}[v^{(1)}, v^{(2)}, v^{(3)}, v^{(4)}] &= \mathcal{F}[v^{(2)}, v^{(1)}, v^{(3)}, v^{(4)}] = \mathcal{F}[v^{(1)}, v^{(2)}, v^{(4)}, v^{(3)}],
\end{aligned}$$

for all $u, v, u^{(n)}, v^{(n)}$, $n = 1,2,3,4$. We also assume that

$$\mathcal{C}_{11}[u^{(1)}, u^{(2)}, v^{(1)}, v^{(2)}]h = \sum_{j,\ell,k} u_j^{(1)} u_{-j}^{(2)} v_\ell^{(1)} v_{-\ell}^{(2)} h_k \, c_{11}(j,\ell,k) \, e^{ik\cdot x}$$

for some coefficient $c_{11}(j,\ell,k)$ to be determined, and similarly for all the other terms. One has

$$\partial_t \begin{pmatrix} w \\ z \end{pmatrix} = (I + \mathcal{M}(u,v)) \begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} + \{\partial_t \mathcal{M}(u,v)\} \begin{pmatrix} u \\ v \end{pmatrix} = (I + \mathcal{K}(u,v)) \begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix}$$

where

$$\mathcal{K}(u,v) := (\Phi^{(5)})'(u,v) - I = \mathcal{M}(u,v) + \mathcal{E}(u,v) \tag{5.3}$$

and, thanks to the previous assumptions,

$$\begin{aligned}
\mathcal{E}(u,v) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} := \{ & 2\mathcal{A}[u,\alpha,u,u] + 2\mathcal{A}[u,u,u,\alpha] + 2\mathcal{B}[u,\alpha,u,v] \\
& + \mathcal{B}[u,u,\alpha,v] + \mathcal{B}[u,u,u,\beta] + 2\mathcal{C}[u,\alpha,v,v] + 2\mathcal{C}[u,u,v,\beta] + \mathcal{D}[\alpha,v,v,v] \\
& + \mathcal{D}[u,\beta,v,v] + 2\mathcal{D}[u,v,v,\beta] + 2\mathcal{F}[v,\beta,v,v] + 2\mathcal{F}[v,v,v,\beta] \} \begin{pmatrix} u \\ v \end{pmatrix}.
\end{aligned} \tag{5.4}$$

The transformed equation is

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = W(u,v)$$

where

$$W(u,v) := (I+\mathcal{K}(u,v))^{-1}X^+(\Phi^{(5)}(u,v)). \tag{5.5}$$

Recalling (4.32), we decompose

$$W(u,v) = \big(1+\mathcal{P}(\Phi^{(5)}(u,v))\big)\big(\mathcal{D}_1(u,v)+X_3^+(u,v)\big)+W_5(u,v)+W_{\geq 7}(u,v), \tag{5.6}$$

where $(1+\mathcal{P}(\Phi^{(5)}))(\mathcal{D}_1+X_3^+)$ gives no contribution to the energy estimate,

$$W_5(u,v) := X_5^+(u,v)+\mathcal{D}_1(\mathcal{M}(u,v)[u,v])-\mathcal{K}(u,v)\mathcal{D}_1(u,v) \tag{5.7}$$

and $W_{\geq 7}(u,v)$ is defined by difference and contains only terms of homogeneity at least seven in (u,v).

We calculate each term of the first component $(W_5)_1$ of W_5. First, one has

$$\begin{aligned}
(W_5)_1(u,v) &= (X_5^+)_1(u,v) - i\Lambda\big(\mathcal{M}_{11}(u,v)u+\mathcal{M}_{12}(u,v)v\big)\\
&\quad - \Big(\mathcal{M}_{11}(u,v)(-i\Lambda u)+\mathcal{M}_{12}(u,v)(i\Lambda v)\Big) - \Big(\mathcal{E}(u,v)\begin{pmatrix}-i\Lambda u\\ i\Lambda v\end{pmatrix}\Big)_1\\
&= (X_5^+)_1(u,v) - 2i\mathcal{M}_{12}(u,v)\Lambda v - \Big(\mathcal{E}(u,v)\begin{pmatrix}-i\Lambda u\\ i\Lambda v\end{pmatrix}\Big)_1.
\end{aligned}$$

Now

$$\begin{aligned}
\Big(\mathcal{E}(u,v)\begin{pmatrix}-i\Lambda u\\ i\Lambda v\end{pmatrix}\Big)_1 &= -2i\mathcal{A}_{11}[u,\Lambda u,u,u]u - 2i\mathcal{A}_{11}[u,u,u,\Lambda u]u - 2i\mathcal{B}_{11}[u,\Lambda u,u,v]u\\
&- i\mathcal{B}_{11}[u,u,\Lambda u,v]u + i\mathcal{B}_{11}[u,u,u,\Lambda v]u - 2i\mathcal{C}_{11}[u,\Lambda u,v,v]u + 2i\mathcal{C}_{11}[u,u,v,\Lambda v]u\\
&- i\mathcal{D}_{11}[\Lambda u,v,v,v]u + i\mathcal{D}_{11}[u,\Lambda v,v,v]u + 2i\mathcal{D}_{11}[u,v,v,\Lambda v]u + 2i\mathcal{F}_{11}[v,\Lambda v,v,v]u\\
&+ 2i\mathcal{F}_{11}[v,v,v,\Lambda v]u - 2i\mathcal{A}_{12}[u,\Lambda u,u,u]v - 2i\mathcal{A}_{12}[u,u,u,\Lambda u]v - 2i\mathcal{B}_{12}[u,\Lambda u,u,v]v\\
&- i\mathcal{B}_{12}[u,u,\Lambda u,v]v + i\mathcal{B}_{12}[u,u,u,\Lambda v]v - 2i\mathcal{C}_{12}[u,\Lambda u,v,v]v + 2i\mathcal{C}_{12}[u,u,v,\Lambda v]v\\
&- i\mathcal{D}_{12}[\Lambda u,v,v,v]v + i\mathcal{D}_{12}[u,\Lambda v,v,v]v + 2i\mathcal{D}_{12}[u,v,v,\Lambda v]v + 2i\mathcal{F}_{12}[v,\Lambda v,v,v]v\\
&+ 2i\mathcal{F}_{12}[v,v,v,\Lambda v]v.
\end{aligned}$$

Thus the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} u_\ell u_{-\ell} u_k e^{ik\cdot x}$ are

$$\begin{aligned}
&Y_{11}^{(4)}[u,u,u,u]u + 2i\mathcal{A}_{11}[u,\Lambda u,u,u]u + 2i\mathcal{A}_{11}[u,u,u,\Lambda u]u\\
&= \sum_{j,\ell,k} u_j u_{-j} u_\ell u_{-\ell} u_k e^{ik\cdot x}\Big(2i(|j|+|\ell|)a_{11}(j,\ell,k) - \frac{i}{64}\Big(\frac{|j|^2|\ell|^2}{|j|+|k|}+\frac{|j|^2|\ell|^2}{|\ell|+|k|}\Big)\Big).
\end{aligned}$$

Hence we choose

$$a_{11}(j,\ell,k) := \frac{|j|^2|\ell|^2}{128(|j|+|\ell|)}\Big(\frac{1}{|j|+|k|}+\frac{1}{|\ell|+|k|}\Big), \tag{5.8}$$

so that $(W_5)_1(u,v)$ does not contain monomials of the type $u_j u_{-j} u_\ell u_{-\ell} u_k e^{ik\cdot x}$.

Next, since $(X_5^+)_1(u,v)$ does not contain monomials $u_j u_{-j} u_\ell v_{-\ell} u_k e^{ik\cdot x}$, we fix

$$\mathcal{B}_{11} = 0, \tag{5.9}$$

so that $(W_5)_1(u,v)$ also does not contain such monomials.

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}$ are

$$\begin{aligned}
&Y_{11}^{(2)}[u,u,v,v]u + 2i\mathcal{C}_{11}[u,\Lambda u,v,v]u - 2i\mathcal{C}_{11}[u,u,v,\Lambda v]u\\
&= \sum_{j,\ell,k} u_j u_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}\Big\{\frac{i}{32}|j|^2|\ell|^2\Big(\frac{-\delta_{|\ell|}^{|k|}(1-\delta_{|j|}^{|k|})}{|j|-|k|} + \frac{1}{|j|+|k|} - \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\\
&\quad + 2ic_{11}(j,\ell,k)(|j|-|\ell|)\Big\}.
\end{aligned}$$

This term can be eliminated for $|j|\neq|\ell|$, while for $|j|=|\ell|$ it cannot be eliminated, and in that case we fix $c_{11}=0$. Thus we choose

$$c_{11}(j,\ell,k) := \frac{1}{64}|j|^2|\ell|^2\Big(\frac{-\delta_{|\ell|}^{|k|}(1-\delta_{|j|}^{|k|})}{|j|-|k|} + \frac{1}{|j|+|k|} - \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\frac{1-\delta_{|j|}^{|\ell|}}{|\ell|-|j|}, \tag{5.10}$$

and the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}$ become

$$\begin{aligned}
&\sum_{\substack{j,\ell,k\\|j|=|\ell|}} u_j u_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}\Big\{\frac{i}{32}|j|^2|\ell|^2\Big(\frac{-\delta_{|\ell|}^{|k|}(1-\delta_{|j|}^{|k|})}{|j|-|k|} + \frac{1}{|j|+|k|} - \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\Big\}\\
&= \frac{i}{32}\sum_{\substack{j,\ell,k\\|j|=|\ell|}} u_j u_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}|j|^2|\ell|^2\Big(\frac{1}{|j|+|k|} - \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big).
\end{aligned}$$

Next, since $(X_5^+)_1(u,v)$ does not contain monomials $u_j v_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}$, we fix

$$\mathcal{D}_{11} = 0, \tag{5.11}$$

so that $(W_5)_1(u,v)$ also does not contain such monomials.

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $v_j v_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}$ are

$$\begin{aligned}
&Y_{11}^{(0)}[v,v,v,v]u - 2i\mathcal{F}_{11}[v,\Lambda v,v,v]u - 2i\mathcal{F}_{11}[v,v,v,\Lambda v]u\\
&= \sum_{j,\ell,k} v_j v_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}\Big\{\frac{i}{64}|j|^2|\ell|^2\Big(-\frac{\delta_{|\ell|}^{|k|}+\delta_{|j|}^{|k|}}{|j|+|\ell|} + \frac{(1-\delta_{|j|}^{|k|})}{|j|-|k|} + \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\\
&\quad - 2if_{11}(j,\ell,k)(|j|+|\ell|)\Big\}.
\end{aligned}$$

Hence we fix

$$f_{11}(j,\ell,k) := \frac{1}{128}\Big(-\frac{\delta_{|\ell|}^{|k|}+\delta_{|j|}^{|k|}}{|j|+|\ell|} + \frac{(1-\delta_{|j|}^{|k|})}{|j|-|k|} + \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\frac{|j|^2|\ell|^2}{|j|+|\ell|}, \tag{5.12}$$

so that $(W_5)_1(u,v)$ does not contain monomials of the type $v_j v_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}$.

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} u_\ell u_{-\ell} v_k e^{ik\cdot x}$ are

$$\begin{aligned}
&Y_{12}^{(4)}[u,u,u,u]v - 2i\mathcal{A}_{12}[u,u,u,u]\Lambda v + 2i\mathcal{A}_{12}[u,\Lambda u,u,u]v + 2i\mathcal{A}_{12}[u,u,u,\Lambda u]v\\
&= \sum_{j,\ell,k} u_j u_{-j} u_\ell u_{-\ell} v_k e^{ik\cdot x}\Big\{\frac{3i}{32}|j||\ell|(|j|+|\ell|) - 2ia_{12}(j,\ell,k)(|k|-|j|-|\ell|)\Big\}.
\end{aligned}$$

Hence we fix

$$a_{12}(j,\ell,k) := \frac{3}{64}|j||\ell|(|j|+|\ell|)\frac{(1-\delta_{|k|}^{|j|+|\ell|})}{|k|-|j|-|\ell|}, \tag{5.13}$$

and the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} u_\ell u_{-\ell} v_k e^{ik\cdot x}$ become

$$\frac{3i}{32}\sum_{\substack{j,\ell,k\\|k|=|j|+|\ell|}} u_j u_{-j} u_\ell u_{-\ell} v_k e^{ik\cdot x}|j||\ell||k|.$$

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} u_\ell v_{-\ell} v_k e^{ik\cdot x}$ are

$$\begin{aligned}
&Y_{12}^{(3)}[u,u,u,v]v - 2i\mathcal{B}_{12}[u,u,u,v]\Lambda v + 2i\mathcal{B}_{12}[u,\Lambda u,u,v]v\\
&\qquad + i\mathcal{B}_{12}[u,u,\Lambda u,v]v - i\mathcal{B}_{12}[u,u,u,\Lambda v]v\\
&\quad = \sum_{j,\ell,k} u_j u_{-j} u_\ell v_{-\ell} v_k e^{ik\cdot x}\Big\{\frac{i}{16}|j|^2|\ell|\Big(\frac{|\ell|\delta_{|\ell|}^{|j|}(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}+6+\frac{|\ell|}{|\ell|+|j|}\\
&\qquad + \frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big) - 2ib_{12}(j,\ell,k)(|k|-|j|)\Big\}.
\end{aligned}$$

Hence we fix

$$b_{12}(j,\ell,k) := \frac{|j|^2|\ell|}{32}\Big(\frac{|\ell|\delta_{|\ell|}^{|j|}(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}+6+\frac{|\ell|}{|\ell|+|j|}+\frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big)\frac{1-\delta_{|j|}^{|k|}}{|k|-|j|}, \tag{5.14}$$

and the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} u_\ell v_{-\ell} v_k e^{ik\cdot x}$ become

$$\begin{aligned}
&\sum_{\substack{j,\ell,k\\|j|=|k|}} u_j u_{-j} u_\ell v_{-\ell} v_k e^{ik\cdot x}\frac{i}{16}|j|^2|\ell|\Big(\frac{|\ell|\delta_{|\ell|}^{|j|}(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}+6+\frac{|\ell|}{|\ell|+|j|}+\frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big)\\
&= \frac{i}{16}\sum_{\substack{j,\ell,k\\|j|=|k|}} u_j u_{-j} u_\ell v_{-\ell} v_k e^{ik\cdot x}|j|^2|\ell|\Big(6+\frac{|\ell|}{|\ell|+|j|}+\frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big).
\end{aligned}$$

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}$ are

$$\begin{aligned}
&Y_{12}^{(2)}[u,u,v,v]v - 2i\mathcal{C}_{12}[u,u,v,v]\Lambda v + 2i\mathcal{C}_{12}[u,\Lambda u,v,v]v - 2i\mathcal{C}_{12}[u,u,v,\Lambda v]v\\
&\quad = \sum_{j,\ell,k} u_j u_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}\Big\{\frac{3i}{16}|j||\ell|(|j|-|\ell|) - 2ic_{12}(j,\ell,k)(|k|-|j|+|\ell|)\Big\}.
\end{aligned}$$

Hence we fix

$$c_{12}(j,\ell,k) := \frac{3}{32}|j||\ell|(|j|-|\ell|)\frac{1-\delta_{|k|}^{|j|-|\ell|}}{|k|-|j|+|\ell|}, \tag{5.15}$$

and the terms in $(W_5)_1(u,v)$ containing the monomials $u_j u_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}$ become

$$\frac{3i}{16}\sum_{\substack{j,\ell,k\\|k|=|j|-|\ell|}} u_j u_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}|j||\ell||k|.$$

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $u_j v_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}$ are

$$
\begin{aligned}
&Y_{12}^{(1)}[u,v,v,v]v - 2i\mathcal{D}_{12}[u,v,v,v]\Lambda v + i\mathcal{D}_{12}[\Lambda u,v,v,v]v \\
&\qquad - i\mathcal{D}_{12}[u,\Lambda v,v,v]v - 2i\mathcal{D}_{12}[u,v,v,\Lambda v]v \\
&= \sum_{j,\ell,k} u_j v_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}\Big\{\frac{i}{16}\sum_{j,\ell,k}|j||\ell|^2\Big(\frac{-|j|\delta_{|j|}^{|\ell|}}{|j|+|k|} - 6 + \frac{|j|(1-\delta_{|j|}^{|\ell|})}{|\ell|-|j|} \\
&\qquad - \frac{|j|}{|\ell|+|j|}\Big) - 2id_{12}(j,\ell,k)(|k|+|\ell|)\Big\}.
\end{aligned}
$$

Hence we fix

$$
d_{12}(j,\ell,k) := \frac{|j||\ell|^2}{32(|k|+|\ell|)}\Big(\frac{-|j|\delta_{|j|}^{|\ell|}}{|j|+|k|} - 6 + \frac{|j|(1-\delta_{|j|}^{|\ell|})}{|\ell|-|j|} - \frac{|j|}{|\ell|+|j|}\Big), \tag{5.16}
$$

so that $(W_5)_1(u,v)$ does not contain monomials of the type $u_j v_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}$.

Next, the terms in $(W_5)_1(u,v)$ containing the monomials $v_j v_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}$ are

$$
\begin{aligned}
&Y_{12}^{(0)}[v,v,v,v]v - 2i\mathcal{F}_{12}[v,v,v,v]\Lambda v - 2i\mathcal{F}_{12}[v,\Lambda v,v,v]v - 2i\mathcal{F}_{12}[v,v,v,\Lambda v]v \\
&= \sum_{j,\ell,k} v_j v_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}\Big\{-\frac{3i}{32}\sum_{j,\ell,k}|j||\ell|(|j|+|\ell|) - 2if_{12}(j,\ell,k)(|k|+|j|+|\ell|)\Big\}.
\end{aligned}
$$

Hence we fix

$$
f_{12}(j,\ell,k) := -\frac{3|j||\ell|(|j|+|\ell|)}{64(|k|+|j|+|\ell|)}, \tag{5.17}
$$

so that $(W_5)_1(u,v)$ does not contain monomials of the type $v_j v_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}$.

Summarizing, it remains

$$
\begin{aligned}
(W_5)_1(u,v) &= \frac{i}{32}\sum_{\substack{j,\ell,k\\|j|=|\ell|}} u_j u_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}|j|^2|\ell|^2\Big(\frac{1}{|j|+|k|} - \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big) \\
&+ \frac{3i}{32}\sum_{\substack{j,\ell,k\\|k|=|j|+|\ell|}} u_j u_{-j} u_\ell u_{-\ell} v_k e^{ik\cdot x}|j||\ell||k| \\
&+ \frac{i}{16}\sum_{\substack{j,\ell,k\\|j|=|k|}} u_j u_{-j} u_\ell v_{-\ell} v_k e^{ik\cdot x}|j|^2|\ell|\Big(6 + \frac{|\ell|}{|\ell|+|j|} + \frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big) \\
&+ \frac{3i}{16}\sum_{\substack{j,\ell,k\\|k|=|j|-|\ell|}} u_j u_{-j} v_\ell v_{-\ell} v_k e^{ik\cdot x}|j||\ell||k|.
\end{aligned} \tag{5.18}
$$

With similar calculations, or deducing the formula from the real structure, the second component $(W_5)_2$ of W_5 is

$$
\begin{aligned}
(W_5)_2(u,v) = &-\frac{i}{32}\sum_{\substack{j,\ell,k\\|j|=|\ell|}} v_j v_{-j} u_\ell u_{-\ell} v_k e^{ik\cdot x}|j|^2|\ell|^2\Big(\frac{1}{|j|+|k|}-\frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\\
&-\frac{3i}{32}\sum_{\substack{j,\ell,k\\|k|=|j|+|\ell|}} v_j v_{-j} v_\ell v_{-\ell} u_k e^{ik\cdot x}|j||\ell||k|\\
&-\frac{i}{16}\sum_{\substack{j,\ell,k\\|j|=|k|}} v_j v_{-j} v_\ell u_{-\ell} u_k e^{ik\cdot x}|j|^2|\ell|\Big(6+\frac{|\ell|}{|\ell|+|j|}+\frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big)\\
&-\frac{3i}{16}\sum_{\substack{j,\ell,k\\|k|=|j|-|\ell|}} v_j v_{-j} u_\ell u_{-\ell} u_k e^{ik\cdot x}|j||\ell||k|. \qquad (5.19)
\end{aligned}
$$

Lemma 5.1. *For all $s \geq 0$, all $(w,z) \in H_0^s(\mathbb{T}^d, c.c.) \cap H_0^{m_0}(\mathbb{T}^d, c.c.)$, one has*

$$
\|W_5(u,v)\|_s \leq C\|u\|_{m_0}^4\|u\|_s,
$$

where C is a universal constant.

Proof. The estimate is deduced from (5.18)-(5.19), using the following bound: if $\alpha, \beta \in \mathbb{Z}^d \setminus \{0\}$, $0 < ||\alpha| - |\beta|| < 1$, then $|\alpha|^2 - |\beta|^2$ is a nonzero integer, $|\alpha| \leq 2|\beta|$, $|\beta| \leq 2|\alpha|$, and

$$
\frac{1}{||\alpha|-|\beta||} = \frac{|\alpha|+|\beta|}{||\alpha|^2-|\beta|^2|} \leq |\alpha|+|\beta| \leq C|\alpha| \leq C'|\beta|. \qquad (5.20)
$$

□

By (5.18)-(5.19), the system for the Fourier coefficients becomes

$$
\begin{aligned}
\partial_t u_k = &-i(1+\mathcal{P})\Big(|k|u_k + \frac{1}{4}\sum_{|j|=|k|} u_j u_{-j}|j|^2 v_k\Big)\\
&+\frac{i}{32}\sum_{\substack{j,\ell\\|j|=|\ell|}} u_j u_{-j} v_\ell v_{-\ell} u_k |j|^2|\ell|^2\Big(\frac{1}{|j|+|k|}-\frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big)\\
&+\frac{3i}{32}\sum_{\substack{j,\ell\\|j|+|\ell|=|k|}} u_j u_{-j} u_\ell u_{-\ell} v_k |j||\ell||k|\\
&+\frac{i}{16}\sum_{\substack{j,\ell\\|j|=|k|}} u_j u_{-j} u_\ell v_{-\ell} v_k |j|^2|\ell|\Big(6+\frac{|\ell|}{|\ell|+|j|}+\frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big)\\
&+\frac{3i}{16}\sum_{\substack{j,\ell\\|j|-|\ell|=|k|}} u_j u_{-j} v_\ell v_{-\ell} v_k |j||\ell||k| + [(W_{\geq 7})_1(u,v)]_k \qquad (5.21)
\end{aligned}
$$

and

$$
\begin{aligned}
\partial_t v_k = i(1+\mathcal{P})\Big(&|k| v_k + \frac{1}{4}\sum_{|j|=|k|} v_j v_{-j} |j|^2 u_k\Big) \\
&- \frac{i}{32}\sum_{\substack{j,\ell \\ |j|=|\ell|}} v_j v_{-j} u_\ell u_{-\ell} v_k |j|^2 |\ell|^2 \Big(\frac{1}{|j|+|k|} - \frac{(1-\delta_{|\ell|}^{|k|})}{|\ell|-|k|}\Big) \\
&- \frac{3i}{32}\sum_{\substack{j,\ell \\ |j|+|\ell|=|k|}} v_j v_{-j} v_\ell v_{-\ell} u_k |j||\ell||k| \\
&- \frac{i}{16}\sum_{\substack{j,\ell \\ |j|=|k|}} v_j v_{-j} v_\ell u_{-\ell} u_k |j|^2 |\ell| \Big(6 + \frac{|\ell|}{|\ell|+|j|} + \frac{|\ell|(1-\delta_{|\ell|}^{|j|})}{|\ell|-|j|}\Big) \\
&- \frac{3i}{16}\sum_{\substack{j,\ell \\ |j|-|\ell|=|k|}} v_j v_{-j} u_\ell u_{-\ell} u_k |j||\ell||k| + [(W_{\geq 7})_2(u,v)]_k
\end{aligned}
\tag{5.22}
$$

where $[(W_{\geq 7})_1(u,v)]_k$ denotes the k-th Fourier coefficient of the first component of $W_{\geq 7}(u,v)$, and similarly for the second component.

Now we prove that the transformation $\Phi^{(5)}$ is bounded and invertible in a ball. Let us begin with estimating the denominators $|k| \pm |j| \pm |\ell|$.

Lemma 5.2. *Let $d \geq 2$, and let $k, j, \ell \in \mathbb{Z}^d \setminus \{0\}$. If $|k| - |j| + |\ell|$ is nonzero, then*

$$
\Big|\frac{1}{|k|-|j|+|\ell|}\Big| \leq C|j|^2|\ell|. \tag{5.23}
$$

If $|k| - |j| - |\ell|$ is nonzero, then

$$
\Big|\frac{1}{|k|-|j|-|\ell|}\Big| \leq C|j||\ell|(|j|+|\ell|). \tag{5.24}
$$

The constant C is universal ($C = 27$ is enough).

Proof. Let $|k| - |j| + |\ell| \neq 0$. If $||k| - |j| + |\ell|| \geq 1$, then (5.23) trivially holds. Thus, assume that

$$
0 < ||k| - |j| + |\ell|| < 1. \tag{5.25}
$$

Since $|j| \geq 1$, it follows that

$$
|k| + |\ell| < |j| + 1 \leq 2|j|. \tag{5.26}
$$

The product

$$
\begin{aligned}
p &:= (|k|+|j|+|\ell|)(|k|+|j|-|\ell|)(|k|-|j|+|\ell|)(|k|-|j|-|\ell|) \\
&= (|k|^2+|j|^2-|\ell|^2)^2 - 4|k|^2|j|^2
\end{aligned}
\tag{5.27}
$$

is an integer. If $p \neq 0$, then $|p| \geq 1$, and, using (5.26),

$$
\begin{aligned}
\Big|\frac{1}{|k|-|j|+|\ell|}\Big| &\leq |(|k|+|j|+|\ell|)(|k|+|j|-|\ell|)(|k|-|j|-|\ell|)| \\
&\leq (3|j|)(3|j|)(3|\ell|) = C|j|^2|\ell|.
\end{aligned}
$$

If $p = 0$, then $|k| + |j| - |\ell| = 0$ or $|k| - |j| - |\ell| = 0$. If $|k| + |j| - |\ell| = 0$, then $|k| - |j| + |\ell| = 2|k| \geq 2$, which contradicts (5.25). If $|k| - |j| - |\ell| = 0$, then $|k| - |j| + |\ell| = 2|\ell| \geq 2$, which also contradicts (5.25). This completes the proof of (5.23).

Now we prove (5.24). Let $|k| - |j| - |\ell| \neq 0$. If $||k| - |j| - |\ell|| \geq 1$, then (5.24) trivially holds. Thus, assume that

$$0 < ||k| - |j| - |\ell|| < 1. \tag{5.28}$$

Then

$$|k| < |j| + |\ell| + 1 \leq 2(|j| + |\ell|).$$

Recalling (5.27), if $p \neq 0$, then $|p| \geq 1$, and

$$\Big|\frac{1}{|k| - |j| - |\ell|}\Big| \leq |(|k| + |j| + |\ell|)(|k| + |j| - |\ell|)(|k| - |j| + |\ell|)| \leq C(|j| + |\ell|)|j||\ell|.$$

If $p = 0$, then $|k| + |j| - |\ell| = 0$ or $|k| - |j| + |\ell| = 0$. If $|k| + |j| - |\ell| = 0$, then $||k| - |j| - |\ell|| = 2|j| \geq 2$, which contradicts (5.28). If $|k| - |j| + |\ell| = 0$, then $||k| - |j| - |\ell|| = 2|\ell| \geq 2$, which also contradicts (5.28). □

Remark 5.3. The bound $|p| \geq 1$ in the proof of Lemma 5.2 is sharp. Indeed, it is enough to show that there are infinitely many choices of $k, j, \ell \in \mathbb{Z}^d \setminus \{0\}$ such that the triple $(|k|^2, |j|^2, |\ell|^2)$ is of the form $(n, n+1, 4n+2)$ for some $n \in \mathbb{N}$. In dimension $d \geq 3$, this is trivial.

In dimension $d = 2$, recall that the set of integers that can be written as the sum of two squares is closed under multiplication, by Brahmagupta's identity

$$(x^2 + y^2)(z^2 + w^2) = (xz + yw)^2 + (xw - yz)^2.$$

Then, it is enough to observe that for $n = 4$ the triple $(n, n+1, 4n+2) = (4, 5, 18) = (2^2 + 0^2, 2^2 + 1^2, 3^2 + 3^2)$ contains only numbers that are the sum of two squares, and that, given any triple $(n, n+1, 4n+2)$ that contains only numbers that are the sum of two squares, the triple $(2n^2 + 2n, 2n^2 + 2n + 1, 4(2n^2 + 2n) + 2)$ has the same property. Indeed, $2n^2 + 2n + 1 = n^2 + (n+1)^2$ and $4(2n^2 + 2n) + 2 = (2n+1)^2 + (2n+1)^2$ are sums of two squares for any $n \in \mathbb{N}$, while $2n^2 + 2n = 2n(n+1)$ is the sum of two squares since it is the product of numbers that are the sum of two squares ($n, n+1$ are sums of two squares by assumption, and $2 = 1^2 + 1^2$).

Lemma 5.4. *For $d \geq 2$, the coefficients $a_{11}, c_{11}, f_{11}, a_{12}, b_{12}, c_{12}, d_{12}, f_{12}$ in* (5.8)-(5.17) *all satisfy the bound*

$$|\textit{coefficient}(k, j, \ell)| \leq C(|j|^4|\ell|^2 + |j|^2|\ell|^4)$$

for some universal constant C. For $d = 1$, they satisfy

$$|\textit{coefficient}(k, j, \ell)| \leq C|j|^2|\ell|^2.$$

Proof. Let $d \geq 2$. The denominators estimated in Lemma 5.2 appear only in a_{12} and c_{12}. The estimate for $|a_{12}|$ directly follows from (5.13) and (5.24). To estimate $|c_{12}|$, for $0 < ||k| - |j| + |\ell|| < 1$ use (5.23) and (5.26), otherwise $|c_{12}| \leq C|j||\ell|(|j| + |\ell|)$. The estimate of a_{11}, f_{12} is trivial. To estimate $c_{11}, f_{11}, b_{12}, d_{12}$, use repeatedly bound (5.20). In dimension $d = 1$ all the estimates are trivial. □

Lemma 5.5. *Let*

$$m_1 := \begin{cases} 1 & \text{if } d = 1, \\ 2 & \text{if } d \geq 2. \end{cases} \tag{5.29}$$

All the operators $\mathcal{G} \in \{\mathcal{A}_{11}, \mathcal{C}_{11}, \mathcal{F}_{11}, \mathcal{A}_{12}, \mathcal{B}_{12}, \mathcal{C}_{12}, \mathcal{D}_{12}, \mathcal{F}_{12}\}$ *satisfy*

$$\|\mathcal{G}[u, v, w, z]h\|_s \leq C\|u\|_{m_1}\|v\|_{m_1}\|w\|_{m_1}\|z\|_{m_1}\|h\|_s \tag{5.30}$$

for all complex functions u, v, w, z, h*, all real* $s \geq 0$*, where* C *is a universal constant.*

Proof. It is an immediate consequence of Lemma 5.4. $\square$

We recall the definition $\|(w, z)\|_s := \|w\|_s = \|z\|_s$ for all pairs $(w, z) = (w, \overline{w}) \in H_0^s(\mathbb{T}^d, c.c.)$ of complex conjugate functions. By (5.2), (5.3), (5.4), we deduce the following estimates.

Lemma 5.6. *For all* $s \geq 0$*, all* $(u, v) \in H_0^{m_1}(\mathbb{T}^d, c.c.)$*,* $(\alpha, \beta) \in H_0^s(\mathbb{T}^d, c.c.)$ *one has*

$$\left\| \mathcal{M}(u, v) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|_s \leq C\|u\|_{m_1}^4 \|\alpha\|_s, \tag{5.31}$$

$$\left\| \mathcal{K}(u, v) \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|_s \leq C\|u\|_{m_1}^3 (\|u\|_{m_1}\|\alpha\|_s + \|u\|_s\|\alpha\|_{m_1}), \tag{5.32}$$

where m_1 *is defined in* (5.29) *and* C *is a universal constant. There exists a universal* $\delta > 0$ *such that, for* $\|u\|_{m_1} < \delta$*, the operator* $(I + \mathcal{K}(u, v)) : H_0^{m_1}(\mathbb{T}^d, c.c.) \to H_0^{m_1}(\mathbb{T}^d, c.c.)$ *is invertible, with inverse*

$$(I + \mathcal{K}(u, v))^{-1} = I - \mathcal{K}(u, v) + \tilde{\mathcal{K}}(u, v), \quad \tilde{\mathcal{K}}(u, v) := \sum_{n=2}^{\infty} (-\mathcal{K}(u, v))^n, \tag{5.33}$$

satisfying

$$\left\| (I + \mathcal{K}(u, v))^{-1} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right\|_s \leq C(\|\alpha\|_s + \|u\|_{m_1}^3 \|u\|_s \|\alpha\|_{m_1}),$$

for all $s \geq 0$.

The nonlinear, continuous map $\Phi^{(5)}$ is invertible in a ball around the origin.

Lemma 5.7. *There exists a universal constant* $\delta > 0$ *such that, for all* $(w, z) \in H_0^{m_1}(\mathbb{T}^d, c.c.)$ *in the ball* $\|w\|_{m_1} \leq \delta$*, there exists a unique* $(u, v) \in H_0^{m_1}(\mathbb{T}^d, c.c.)$ *such that* $\Phi^{(5)}(u, v) = (w, z)$*, with* $\|u\|_{m_1} \leq 2\|w\|_{m_1}$*. If, in addition,* $w \in H_0^s$ *for some* $s > m_1$*, then* u *also belongs to* H_0^s*, and* $\|u\|_s \leq 2\|w\|_s$*. This defines the continuous inverse map* $(\Phi^{(5)})^{-1} : H_0^s(\mathbb{T}^d, c.c.) \cap \{\|w\|_{m_1} \leq \delta\} \to H_0^s(\mathbb{T}^d, c.c.)$.

Proof. Using the estimates of Lemma 5.6, the proof of Lemma 5.7 is a straightforward adaptation of the proof of Lemma 4.3 in [4]. $\square$

We estimate the remainder $W_{\geq 7}(u,v)$. By (5.6) (which is the definition of $W_{\geq 7}(u,v)$) and (5.7), (5.5), (4.32), (5.1), we calculate

$$\begin{aligned}
W_{\geq 7}(u,v) &= \tilde{\mathcal{K}}(u,v)[1+\mathcal{P}(\Phi^{(5)}(u,v))]\mathcal{D}_1(u,v) \\
&\quad + (-\mathcal{K}(u,v)+\tilde{\mathcal{K}}(u,v))[1+\mathcal{P}(\Phi^{(5)}(u,v))]\mathcal{D}_1(\mathcal{M}(u,v)[u,v]) \\
&\quad - \mathcal{K}(u,v)\mathcal{P}(\Phi^{(5)}(u,v))\mathcal{D}_1(u,v) \\
&\quad + \mathcal{P}(\Phi^{(5)}(u,v))\mathcal{D}_1(\mathcal{M}(u,v)[u,v]) \\
&\quad + (-\mathcal{K}(u,v)+\tilde{\mathcal{K}}(u,v))[1+\mathcal{P}(\Phi^{(5)}(u,v))]X_3^+(u,v) \\
&\quad + (-\mathcal{K}(u,v)+\tilde{\mathcal{K}}(u,v))X_5^+(u,v) \\
&\quad + (I+\mathcal{K}(u,v))^{-1}[1+\mathcal{P}(\Phi^{(5)}(u,v))][X_3^+(\Phi^{(5)}(u,v)) - X_3^+(u,v)] \\
&\quad + (I+\mathcal{K}(u,v))^{-1}[X_5^+(\Phi^{(5)}(u,v)) - X_5^+(u,v)] \\
&\quad + (I+\mathcal{K}(u,v))^{-1}X_{\geq 7}^+(\Phi^{(5)}(u,v)), \qquad (5.34)
\end{aligned}$$

where $\tilde{\mathcal{K}}(u,v)$ is defined in (5.33). The only unbounded operator appearing in (5.34) is $\mathcal{D}_1$. We rewrite the terms containing $\mathcal{D}_1$ by using the "homological equation" (5.7) (which is, in short, $\mathcal{D}_1\mathcal{M} - \mathcal{K}\mathcal{D}_1 = W_5 - X_5^+$) and the fact that the multiplication by $\mathcal{P}(\Phi^{(5)}(u,v))$ commutes with $\mathcal{K}(u,v)$, because $\mathcal{P}(\Phi^{(5)}(u,v))$ is a real scalar function of time only. Thus, omitting to write (u,v) everywhere, the first two terms in (5.34) become

$$\begin{aligned}
&\tilde{\mathcal{K}}(1+\mathcal{P}(\Phi^{(5)}))\mathcal{D}_1 + (-\mathcal{K}+\tilde{\mathcal{K}})(1+\mathcal{P}(\Phi^{(5)}))\mathcal{D}_1\mathcal{M} \\
&\quad = (1+\mathcal{P}(\Phi^{(5)}))\Big(\sum_{n=2}^{\infty}(-\mathcal{K})^n\mathcal{D}_1 + \sum_{n=1}^{\infty}(-\mathcal{K})^n\mathcal{D}_1\mathcal{M}\Big) \\
&\quad = (1+\mathcal{P}(\Phi^{(5)}))\sum_{n=1}^{\infty}(-\mathcal{K})^n(-\mathcal{K}\mathcal{D}_1 + \mathcal{D}_1\mathcal{M}) \\
&\quad = (1+\mathcal{P}(\Phi^{(5)}))(-\mathcal{K}+\tilde{\mathcal{K}})(W_5 - X_5^+).
\end{aligned}$$

Therefore (5.34) becomes

$$\begin{aligned}
W_{\geq 7}(u,v) &= [1+\mathcal{P}(\Phi^{(5)}(u,v))](-\mathcal{K}(u,v)+\tilde{\mathcal{K}}(u,v))(W_5(u,v) - X_5^+(u,v)) \\
&\quad + \mathcal{P}(\Phi^{(5)}(u,v))(W_5(u,v) - X_5^+(u,v)) \\
&\quad + (-\mathcal{K}(u,v)+\tilde{\mathcal{K}}(u,v))[1+\mathcal{P}(\Phi^{(5)}(u,v))]X_3^+(u,v) \\
&\quad + (-\mathcal{K}(u,v)+\tilde{\mathcal{K}}(u,v))X_5^+(u,v) \\
&\quad + (I+\mathcal{K}(u,v))^{-1}[1+\mathcal{P}(\Phi^{(5)}(u,v))][X_3^+(\Phi^{(5)}(u,v)) - X_3^+(u,v)] \\
&\quad + (I+\mathcal{K}(u,v))^{-1}[X_5^+(\Phi^{(5)}(u,v)) - X_5^+(u,v)] \\
&\quad + (I+\mathcal{K}(u,v))^{-1}X_{\geq 7}^+(\Phi^{(5)}(u,v)). \qquad (5.35)
\end{aligned}$$

Lemma 5.8. *There exist universal constants $\delta > 0$, $C > 0$ such that, for all $s \geq 0$, for all $(u,v) \in H_0^{m_1}(\mathbb{T}^d, c.c.) \cap H_0^s(\mathbb{T}^d, c.c.)$ in the ball $\|u\|_{m_1} \leq \delta$, one has*

$$\|W_{\geq 7}(u,v)\|_s \leq C\|u\|_{m_1}^6\|u\|_s.$$

Proof. Use formula (5.35) and Lemmas 4.6, 4.8, 5.1, 5.6, 5.7. □

Energy estimate. By (5.6), the energy estimate for the system $\partial_t(u,v) = W(u,v)$ on the real subspace $\{v = \bar{u}\}$ becomes

$$\partial_t(\|u\|_s^2) = \langle \Lambda^s \partial_t u, \Lambda^s v \rangle + \langle \Lambda^s u, \Lambda^s \partial_t v \rangle = Z_6(u) + Z_{\geq 8}(u) \tag{5.36}$$

where

$$\begin{aligned} Z_6(u) &:= \langle \Lambda^s (W_5)_1(u,v), \Lambda^s v \rangle + \langle \Lambda^s u, \Lambda^s (W_5)_2(u,v) \rangle, \\ Z_{\geq 8}(u) &:= \langle \Lambda^s (W_{\geq 7})_1(u,v), \Lambda^s v \rangle + \langle \Lambda^s u, \Lambda^s (W_{\geq 7})_2(u,v) \rangle, \end{aligned}$$

because the term $\big(1 + \mathcal{P}(\Phi^{(5)}(u,v))\big)\big(\mathcal{D}_1(u,v) + X_3^+(u,v)\big)$ gives zero contribution. By Lemma 5.8, one has

$$|Z_{\geq 8}(u)| \leq C \|u\|_{m_1}^6 \|u\|_s^2.$$

By (5.18)-(5.19), we calculate

$$Z_6(u) = \frac{3i}{32} \sum_{\substack{j,\ell,k \\ |k|=|j|+|\ell|}} (u_j u_{-j} u_\ell u_{-\ell} v_k v_{-k} - v_j v_{-j} v_\ell v_{-\ell} u_k u_{-k}) |j||\ell||k|^{1+2s} \tag{5.37}$$

$$+ \frac{3i}{16} \sum_{\substack{j,\ell,k \\ |k|=|j|-|\ell|}} (u_j u_{-j} v_\ell v_{-\ell} v_k v_{-k} - v_j v_{-j} u_\ell u_{-\ell} u_k u_{-k}) |j||\ell||k|^{1+2s}, \tag{5.38}$$

which is the sum of the second and the fourth sums in both $(W_5)_1$ and $(W_5)_2$, because the first and third sums in $(W_5)_1$ and $(W_5)_2$ cancel out. Then, we note that the sum over $|k| = |j| - |\ell|$ in (5.38), namely $|j| = |k| + |\ell|$, becomes, after renaming the indices, a sum over the same set of indices as the sum in (5.37). Hence

$$Z_6(u) = \frac{3i}{32} \sum_{\substack{j,\ell,k \\ |k|=|j|+|\ell|}} (u_j u_{-j} u_\ell u_{-\ell} v_k v_{-k} - v_j v_{-j} v_\ell v_{-\ell} u_k u_{-k}) |j||\ell||k| \big(|k|^{2s} - 2|j|^{2s}\big),$$

namely, symmetrizing $j \leftrightarrow \ell$,

$$Z_6(u) = \frac{3i}{32} \sum_{\substack{j,\ell,k \\ |k|=|j|+|\ell|}} (u_j u_{-j} u_\ell u_{-\ell} v_k v_{-k} - v_j v_{-j} v_\ell v_{-\ell} u_k u_{-k}) |j||\ell||k| \big(|k|^{2s} - |j|^{2s} - |\ell|^{2s}\big). \tag{5.39}$$

For $s = \frac{1}{2}$, one has $|k|^{2s} - |j|^{2s} - |\ell|^{2s} = |k| - |j| - |\ell| = 0$ over the sum, and therefore $Z_6(u)$ vanishes for $s = \frac{1}{2}$. Hence $\langle \Lambda u, v \rangle$ is a prime integral up to homogeneity order 8, namely

$$|\partial_t(\|u\|_{\frac{1}{2}}^2)| = |\partial_t \langle \Lambda u, v \rangle| \leq C \|u\|_{m_1}^6 \|u\|_{\frac{1}{2}}^2.$$

This is not surprising, since $s = \frac{1}{2}$ in (5.36) corresponds to the norm in the energy space $H^1 \times L^2$ of the original variables, and that norm is controlled by the Hamiltonian.

For $s \neq \frac{1}{2}$, in general the term $Z_6(u)$ is not zero. For example, for $s = 1$ one has $|k|^{2s} - |j|^{2s} - |\ell|^{2s} = (|j| + |\ell|)^2 - |j|^2 - |\ell|^2 = 2|j||\ell|$.

Spheres in Fourier space. We observe that the system (or some relevant aspects of it concerning the evolution of Sobolev norms) can be described by taking sums over all frequencies $k \in \mathbb{Z}^d$ with a fixed (Euclidean) length $|k| = \lambda$. For each λ in the set

$$\Gamma := \{|k| : k \in \mathbb{Z}^d, \ k \neq 0\} \subset [1, \infty), \tag{5.40}$$

let

$$S_\lambda := \sum_{k:|k|=\lambda} |u_k|^2 = \sum_{k:|k|=\lambda} u_k v_{-k}, \quad B_\lambda := \sum_{k:|k|=\lambda} u_k u_{-k},$$

so that

$$\overline{B_\lambda} = \sum_{k:|k|=\lambda} v_k v_{-k}, \qquad \|u\|_s^2 = \sum_{\lambda\in\Gamma} \lambda^{2s} S_\lambda.$$

For each $\lambda \in \Gamma$, $S_\lambda \geq 0$ and $B_\lambda \in \mathbb{C}$. By (5.21)-(5.22), neglecting the terms from $W_{\geq 7}$, one has

$$\partial_t S_\lambda = \frac{3i}{32} \sum_{\substack{\alpha,\beta\in\Gamma \\ \alpha+\beta=\lambda}} (B_\alpha B_\beta \overline{B_\lambda} - \overline{B_\alpha}\,\overline{B_\beta} B_\lambda)\alpha\beta\lambda + \frac{3i}{16} \sum_{\substack{\alpha,\beta\in\Gamma \\ \alpha-\beta=\lambda}} (B_\alpha \overline{B_\beta}\,\overline{B_\lambda} - \overline{B_\alpha} B_\beta B_\lambda)\alpha\beta\lambda \tag{5.41}$$

and

$$\begin{aligned} \partial_t B_\lambda = {} & -2i(1+\mathcal{P})\Big(\lambda + \frac{1}{4}\lambda^2 S_\lambda\Big) B_\lambda + \frac{i}{16} \sum_{\alpha\in\Gamma} |B_\alpha|^2 B_\lambda \alpha^2 \Big(\frac{1}{\alpha+\lambda} - \frac{1-\delta_\alpha^\lambda}{\alpha-\lambda}\Big) \\ & + \frac{3i}{16} \sum_{\substack{\alpha,\beta\in\Gamma \\ \alpha+\beta=\lambda}} B_\alpha B_\beta S_\lambda \lambda\alpha\beta + \frac{i}{8} \sum_{\alpha\in\Gamma} S_\alpha S_\lambda B_\lambda \lambda^2 \alpha \Big(6 + \frac{\alpha}{\alpha+\lambda} + \frac{\alpha(1-\delta_\alpha^\lambda)}{\alpha-\lambda}\Big) \\ & + \frac{3i}{8} \sum_{\substack{\alpha,\beta\in\Gamma \\ \alpha-\beta=\lambda}} B_\alpha \overline{B_\beta} S_\lambda \alpha\beta\lambda. \end{aligned} \tag{5.42}$$

Equations (5.41)-(5.42) form a closed system in the variables $(S_\lambda, B_\lambda)_{\lambda\in\Gamma}$. They play the role of an "effective equation" for the dynamics of the Kirchhoff equation. This will be the starting point for further analysis in the forthcoming paper [5].

References

[1] A. Arosio, *Averaged evolution equations. The Kirchhoff string and its treatment in scales of Banach spaces*, in: 2nd Workshop on functional-analytic methods in complex analysis (Trieste, 1993), World Scientific, Singapore.

[2] A. Arosio, S. Panizzi, *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc. 348 (1996), 305-330.

[3] P. Baldi, *Periodic solutions of forced Kirchhoff equations*, Ann. Sc. Norm. Sup. Pisa, Cl. Sci. (5), Vol. VIII (2009), 117-141.

[4] P. Baldi, E. Haus, *On the existence time for the Kirchhoff equation with periodic boundary conditions*, Nonlinearity 33 (2020), no. 1, 196-223.

[5] P. Baldi, E. Haus, *Longer lifespan for many solutions of the Kirchhoff equation*, preprint (arXiv:2007.03543).

[6] D. Bambusi, *Galerkin averaging method and Poincaré normal form for some quasi-linear PDEs*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4 (2005), 669-702.

[7] S.N. Bernstein, *Sur une classe d'équations fonctionnelles aux dérivées partielles*, Izv. Akad. Nauk SSSR Ser. Mat. 4 (1940), 17-26.

[8] M. Berti, J.-M. Delort, *Almost global solutions of capillary-gravity water waves equations on the circle.* Lecture Notes Unione Mat. Ital. 24, Springer, 2018.

[9] M. Berti, R. Feola, F. Pusateri, *Birkhoff normal form and long time existence for periodic gravity water waves*, preprint (arXiv:1810.11549).

[10] M. Berti, R. Feola, F. Pusateri, *Birkhoff normal form for gravity water waves*, Water Waves (2020), online first.

[11] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, T. Tao, *Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation*, Invent. Math. 181 (2010), 39-113.

[12] L. Corsi, R. Montalto, *Quasi-periodic solutions for the forced Kirchhoff equation on* $\mathbb{T}^d$, Nonlinearity 31 (2018), no. 11, 5075-5109.

[13] W. Craig, C. Sulem, *Mapping properties of normal forms transformations for water waves*, Boll. Unione Mat. Ital. 9 (2016), 289-318.

[14] Craig W., Worfolk P., *An integrable normal form for water waves in infinite depth.* Phys. D, 84 (1995), no. 3-4, 513-531.

[15] P. D'Ancona, S. Spagnolo, *A class of nonlinear hyperbolic problems with global solutions*, Arch. Rational Mech. Anal. 124 (1993), 201-219.

[16] J.-M. Delort, *Long-time Sobolev stability for small solutions of quasi-linear Klein-Gordon equations on the circle*, Trans. Amer. Math. Soc. 361 (2009), no. 8, 4299-4365.

[17] J.-M. Delort, *A quasi-linear Birkhoff normal forms method. Application to the quasi-linear Klein-Gordon equation on* $\mathbb{S}^1$, Astérisque 341 (2012).

[18] R. W. Dickey, *Infinite systems of nonlinear oscillation equations related to the string*, Proc. Amer. Math. Soc. 23 (1969), 459-468.

[19] R. Feola, F. Iandoli, *Long time existence for fully nonlinear NLS with small Cauchy data on the circle*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (2020), published online.

[20] B. Grébert, L. Thomann, *Resonant dynamics for the quintic nonlinear Schrödinger equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), 455-477.

[21] J.M. Greenberg, S.C. Hu, *The initial value problem for a stretched string*, Quart. Appl. Math. 38 (1980/81), 289-311.

[22] M. Guardia, Z. Hani, E. Haus, A. Maspero, M. Procesi, *Strong nonlinear instability and growth of Sobolev norms near quasiperiodic finite-gap tori for the 2D cubic NLS equation*, J. Eur. Math. Soc. (JEMS), accepted (arXiv:1810.03694).

[23] M. Guardia, E. Haus, M. Procesi, *Growth of Sobolev norms for the analytic NLS on* $\mathbb{T}^2$, Adv. Math. 301 (2016), 615-692.

[24] M. Guardia, V. Kaloshin, *Growth of Sobolev norms in the cubic defocusing nonlinear Schrödinger equation*, J. Eur. Math. Soc. (JEMS) 17 (2015), 71-149.

[25] E. Haus, M. Procesi, *Growth of Sobolev norms for the quintic NLS on* $\mathbb{T}^2$, Anal. PDE 8 (2015), 883-922.

[26] E. Haus, M. Procesi, *KAM for beating solutions of the quintic NLS*, Comm. Math. Phys. 354 (2017), 1101-1132.

[27] E. Haus, L. Thomann, *Dynamics on resonant clusters for the quintic nonlinear Schrödinger equation*, Dyn. Partial Differ. Equ. 10 (2013), 157-169.

[28] M. Ifrim, D. Tataru, *The lifespan of small data solutions in two dimensional capillary water waves*, Arch. Ration. Mech. Anal. 225 (2017), no. 3, 1279-1346.

[29] A.D. Ionescu, F. Pusateri, *Global Regularity for 2d Water Waves with Surface Tension*, Mem. Amer. Math. Soc. 256 (2018), no. 1227.

[30] A.D. Ionescu, F. Pusateri, *Long-time existence for multi-dimensional periodic water waves*, Geom. Funct. Anal. 29 (2019), 811-870.

[31] G. Kirchhoff, *Vorlesungen über mathematische Physik: Mechanik*, ch.29, Teubner, Leipzig, 1876.

[32] J.L. Lions, *On some questions in boundary value problems of mathematical physics*, in: Contemporary developments in continuum mechanics and PDE's, G.M. de la Penha & L.A. Medeiros eds., North-Holland, Amsterdam, 1978.

[33] T. Matsuyama, M. Ruzhansky, *Global well-posedness of the Kirchhoff equation and Kirchhoff systems*, Analytic methods in interdisciplinary applications, 81-96, Springer Proc. Math. Stat., 116, Springer, Cham, 2015.

[34] R. Montalto, *Quasi-periodic solutions of forced Kirchhoff equation*, NoDEA Nonlinear Differential Equations Appl. 24 (2017), Art. 9.

[35] S. Spagnolo, *The Cauchy problem for Kirchhoff equations*, Rend. Sem. Mat. Fis. Milano 62 (1994), 17-51.

Pietro Baldi
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università di Napoli Federico II
Via Cintia, Monte S. Angelo
80126 Napoli
pietro.baldi@unina.it

Emanuele Haus
Dipartimento di Matematica e Fisica
Università Roma Tre
Largo San Leonardo Murialdo 1
00146 Roma
ehaus@mat.uniroma3.it