

NON-EXISTENCE OF *THETA*-SHAPED SELF-SIMILARLY SHRINKING NETWORKS MOVING BY CURVATURE

PIETRO BALDI, EMANUELE HAUS, AND CARLO MANTEGAZZA

ABSTRACT. We prove that there are no networks homeomorphic to the Greek “Theta” letter (a double cell) embedded in the plane with two triple junctions with angles of 120 degrees, such that under the motion by curvature they are self-similarly shrinking. This fact completes the classification of the self-similarly shrinking networks in the plane with at most two triple junctions, see [2, 5, 10, 25].

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1. INTRODUCTION

Recently, the problem of the evolution by curvature of a network of curves in the plane got the interest of several authors [3, 10, 17, 18, 21, 22, 24–26]. It is well known that, after the work of Huisken [13] in the smooth case of the hypersurfaces in the Euclidean space and of Ilmanen [15, 16] in the more general weak settings of varifolds, that a suitable sequence of rescalings of the *subsets* of \mathbb{R}^n which are evolving by mean curvature, approaching a singular time of the flow, converges to a so called “blow-up limit” set which, letting it flow again by mean curvature, simply moves by homothety, precisely, it shrinks down self-similarly toward the origin of the Euclidean space.

This procedure and the classification of these special sets (possibly under some hypotheses), called *shrinkers*, is a key point in understanding the asymptotic behavior of the flow at a singular time.

In the special situation of a network of curves in the plane moving by curvature (see [20, 21], for instance), it is easy to see that every C^2 curve $\gamma : I \rightarrow \mathbb{R}^2$ of such a limit shrinking network must satisfy the following “structural” equation (which is actually an ODE for γ)

$$\bar{k} + \gamma^\perp = 0, \tag{1.1}$$

where \bar{k} is the vector curvature of the curve at the point γ and γ^\perp denotes the normal component of the position vector γ . Introducing an arclength parameter s on the curve γ , we have a unit tangent vector field $\tau = \frac{d}{ds}\gamma$, a unit normal vector field ν which is the counterclockwise rotation of $\pi/2$ in \mathbb{R}^2 of the vector τ and the curvature vector given by $\bar{k} = k\nu = \frac{d^2}{ds^2}\gamma$, where k is then simply the curvature of γ . With these notations, the above equation can be rewritten as

$$k + \langle \gamma | \nu \rangle = 0. \tag{1.2}$$

It is known, by the work of Abresch–Langer [1] and independently of Epstein–Weinstein [7], that the only complete, embedded shrinking curves in \mathbb{R}^2 are the lines through the origin and the unit circle.

Dealing with the motion by curvature of networks of curves in the plane, it is natural (for “energy reasons”) to consider networks with only triple junctions and such that the three concurring curves (which are C^∞) form three angles of 120 degrees between each other (“Herring” condition); such networks are called *regular*. Analogously, a shrinker is called regular if it satisfies such geometric condition.

If one consider networks with only one triple junction, the only complete, embedded, connected, regular shrinkers are given (up to a rotation) by the “standard triod” and the “Brakke spoon” (first described in [4]) as in the following figures.

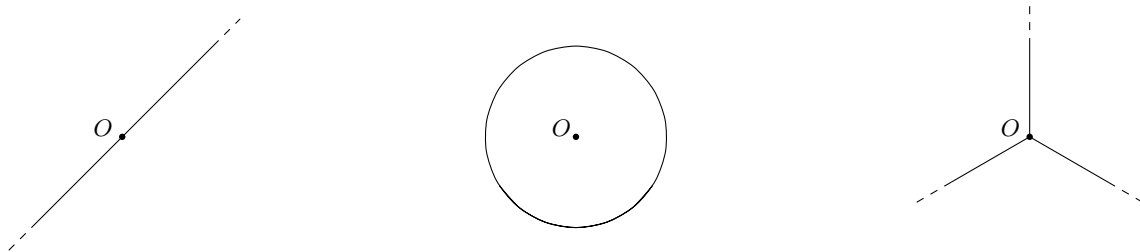


FIGURE 1. Easy examples of regular shrinkers: a line from the origin, the unit circle \mathbb{S}^1 and an unbounded triod composed of three halflines from the origin meeting at 120 degrees, called standard triod.

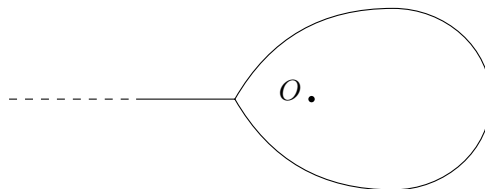


FIGURE 2. A less easy example of a regular shrinker: a Brakke spoon.

About shrinkers with two triple junctions, after the contributions in [5, 10, 25], it can be shown that there are only two possible topological shapes for a complete embedded, regular shrinker: one is the “lens/fish” shape and the other is the shape of the Greek “theta” letter (or “double cell”), as in the next figure (see [2], for more details).

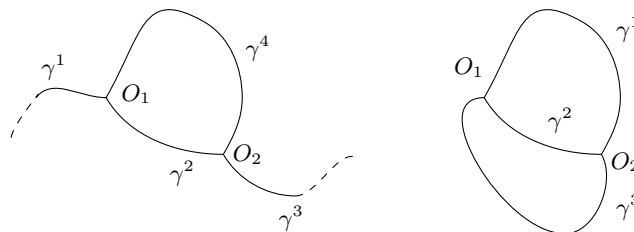


FIGURE 3. A lens/fish-shaped and a Θ -shaped network.

It is well known that there exist unique (up to a rotation) lens-shaped or fish-shaped, embedded, regular shrinkers which are symmetric with respect to a line through the origin of \mathbb{R}^2 (see [5, 25]). It was instead unknown whether regular Θ -shaped shrinkers (or simply Θ -shrinkers) exist, with numerical evidence in favor of the conjecture of non-existence (see [10]). In this paper we are going to show that this is actually the case.

Theorem 1.1. *There are no regular Θ -shrinkers.*

The main motivation for this problem is given by the fact that for an evolving network with at most *two* triple junctions the so called *multiplicity-one conjecture* holds (see [20]), saying that any limit shrinker of a sequence of rescalings of the network at different times is again a “genuine” embedded network without “double” or “multiple” curves (curves that in such convergence go to coincide). This is a key point in singularity analysis (in general, for mean curvature flow), together with the classification of these limit shrinkers, which is complete after our result Theorem 1.1, for such “low complexity” networks, thus leading to a full description of their motion in [19].



FIGURE 4. A lens-shaped and a fish-shaped shrinker.

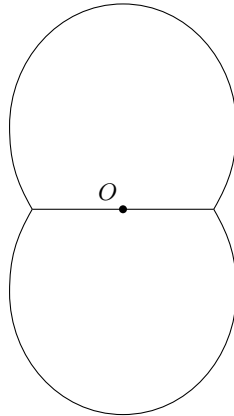


FIGURE 5. A hypothetical Θ -shrinker.

The line to show Theorem 1.1 will be first looking at the properties that any hypothetical Θ -shrinker must satisfy and study the possible geometric structures that *a priori* it could have (Sections 2 and 3). Such analysis will reduce the proof of non-existence to show that a certain parametric integral is always smaller than $\pi/2$, for every value of the parameter. Section 4 is devoted to show such an estimate, mixing some approximation techniques and numerical computations based on *interval arithmetic*.

1.1. Interval arithmetic. When performing standard analytical computations on a machine, the computer routines repeatedly introduce unpleasant errors. This cannot be avoided since, for instance:

- the real numbers that a machine can handle exactly in floating-point binary representation are only a finite subset of the rationals;
- transcendental functions cannot be computed exactly and have to be replaced by an approximation based on a finite Taylor expansion.

While these limitations make it impossible to get an exact result from – basically – any numerical computation on a machine, it is however possible to guarantee that the value of a given analytical expression is included between two values that can be represented *exactly* by the computer, providing rigorous upper and lower bounds.

The term “interval arithmetic” refers to the arithmetic of quantities whose value lies within a known interval, although the exact value of such quantities is not known. The simple considerations presented above clearly demonstrate why the theory and implementation of interval arithmetic is a fundamental tool for making rigorous computer-assisted proofs in mathematical analysis.

Remark 1.2. Interval arithmetic for rigorous computer-assisted proofs has been used in several areas of mathematics (see, for example, the reviews [23] and [8]). See [9] for documentation about correct rounding of elementary functions. Note that all the computations of the present paper that we made with the computer involve only elementary functions (i.e. $x + y$, $x \cdot y$, x^2 , $1/x$, \sqrt{x} , $\exp(x)$, $\log(x)$, $\arcsin(x)$) and constants (π) whose correct rounding is guaranteed by the IEEE Standard for Interval Arithmetic, IEEE 1788–2015 [14]. To perform interval arithmetic, we used the software GNU Octave 4.0.0 [6] with the package *interval* [11], which is conforming (see [12]) to the standard IEEE 1788–2015.

We give a basic example of the use of *interval arithmetic*: to deal with the constant $\sqrt{3}$, we write the command “sqrt(infsup(3))”: writing “infsup(3)” the integer number 3 is transformed into the interval

$[3, 3]$ (which is just the singleton $\{3\}$ since the integer number 3 is exactly representable by the computer), then the function “square root” applies to the interval $[3, 3]$ and it gives a correct rounding of the exact mathematical value of $\sqrt{3}$, namely the output is an interval $[a, b]$ such that $a^2 < 3 < b^2$ and a, b are exactly representable in binary form.

The codes we have used are in Section A.

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2. BASIC PROPERTIES OF SHRINKING CURVES AND NOTATION

Consider a shrinking curve $\gamma : I \rightarrow \mathbb{R}^2$ parametrized in arclength s , where $I \subset \mathbb{R}$ is an interval. We denote with $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ the counterclockwise rotation of 90 degrees. Then, the relation

$$\gamma_{ss} = \frac{d^2\gamma}{ds^2} = k\nu = -\langle \gamma | \nu \rangle \nu = -\left\langle \gamma \mid R\left(\frac{d\gamma}{ds}\right) \right\rangle \nu$$

gives an ODE satisfied by γ . It follows that the curve is smooth and it is not difficult to see that for every point $x_0 \in \mathbb{R}^2$ and unit velocity vector τ_0 , there exists a unique shrinking curve (solution of such ODE) parametrized in arclength, passing at $s = 0$ through the point x_0 with velocity τ_0 , defined for all $s \in \mathbb{R}$.

Differentiating in arclength the equation $k = -\langle \gamma | \nu \rangle$, we get the ODE for the curvature $k_s = k\langle \gamma | \tau \rangle$. Suppose that at some point $k = 0$, then it must also hold $k_s = 0$ at the same point, hence, by the uniqueness theorem for ODEs we conclude that k is identically zero and we are dealing with a line L which, as $\langle x | \nu \rangle = 0$ for every $x \in L$, must contain the origin of \mathbb{R}^2 .

So we suppose that k is always non-zero and, by looking at the structural equation $k + \langle \gamma | \nu \rangle = 0$, we can see that the curve is then strictly convex with respect to the origin of \mathbb{R}^2 . Another consequence (by the uniqueness theorem for ODE) is that the curve must be symmetric with respect to any critical point (maximum or minimum) of its curvature function. Notice that if the curve is not a piece of a circle, they are all non-degenerate and isolated (if the curve has bounded length, their number is finite).

Computing the derivative of $|\gamma|^2$,

$$\frac{d|\gamma|^2}{ds} = 2\langle \gamma | \tau \rangle = 2k_s/k = 2\frac{d \log k}{ds}$$

we get $k = Ce^{|\gamma|^2/2}$ for some constant $C \in \mathbb{R}$, that is, the quantity

$$\mathcal{E} = \mathcal{E}(\gamma) := ke^{-|\gamma|^2/2}, \quad (2.1)$$

that we call *Energy*, is constant along the curve. Equivalently, $\langle \gamma | \nu \rangle e^{-|\gamma|^2/2}$ is constant. A solution γ has positive energy if $k > 0$, so that γ runs counterclockwise around the origin, γ has negative energy if $k < 0$, so that γ runs clockwise around the origin, γ has energy zero if $k = 0$, so that γ is a piece of a straight line through the origin.

We consider now a new coordinate $\theta = \arccos \langle e_1 | \nu \rangle$; this can be done for the whole curve as we know that it is convex (obviously, θ is only locally continuous, since it “jumps” after a complete round).

Differentiating with respect to the arclength parameter we have $\frac{d\theta}{ds} = k$ and

$$k_\theta = k_s/k = \langle \gamma | \tau \rangle \quad k_{\theta\theta} = \frac{1}{k} \frac{dk_\theta}{ds} = \frac{1 + k\langle \gamma | \nu \rangle}{k} = \frac{1}{k} - k. \quad (2.2)$$

Multiplying both sides of the last equation by $2k_\theta$ we get $\frac{d}{d\theta} [k_\theta^2 + k^2 - \log k^2] = 0$, that is, the quantity

$$E := k_\theta^2 + k^2 - \log k^2$$

is constant along all the curve. Notice that such quantity E cannot be less than 1 (if $k \neq 0$), moreover, if $E = 1$ we have that k^2 must be constant and equal to one along the curve, which consequently must be a piece of the unit circle centered at the origin of \mathbb{R}^2 .

As $E \geq 1$, it follows that k^2 is uniformly bounded from above and away from zero, hence, recalling that $k = \mathcal{E}e^{|\gamma|^2/2}$, the curve γ is contained in a ball of \mathbb{R}^2 (and it is outside some small ball around the origin).

Since we are interested in the curves of a non-trivial connected, compact (Θ -shaped), regular network, there will be no unbounded lines or complete circles and all the curves of the network will be images of a closed bounded interval, once parametrized in arclength.

Resuming, either γ is a segment or $k^2 > 0$, the equations (2.2) hold, the Energy $\mathcal{E} = ke^{-|\gamma|^2/2}$ and the quantity $E = k_\theta^2 + k^2 - \log k^2 \geq 1$ are constant along the curve, where $\theta = \arccos \langle e_1 | \nu \rangle$. Moreover, the full curve γ is symmetric with respect to the critical points of the curvature (by the uniqueness of solutions of ODEs), hence the curvature $k(\theta)$ is oscillating between its maximum and its minimum.

Suppose now that $k_{\min} < k_{\max}$ are these two consecutive critical values of k . It follows that they are two distinct positive zeroes of the function $k_\theta^2 = E + \log k^2 - k^2$, when $E > 1$, with $0 < k_{\min} < 1 < k_{\max}$. We have then that the change $\Delta\theta$ in the angle θ along the piece of curve delimited by two consecutive points where the curvature assumes the values k_{\min} and k_{\max} , is given by the integral

$$\Delta\theta = I(E) = \int_{k_{\min}}^{k_{\max}} \frac{dk}{\sqrt{E - k^2 + \log k^2}}. \quad (2.3)$$

Proposition 2.1 (Abresch and Langer [1]). *The function $I : (1, +\infty) \rightarrow \mathbb{R}$ satisfies*

- (1) $\lim_{E \rightarrow 1^+} I(E) = \pi/\sqrt{2}$,
- (2) $\lim_{E \rightarrow +\infty} I(E) = \pi/2$,
- (3) $I(E)$ is monotone decreasing.

As a consequence $I(E) > \pi/2$.

We write now the curve γ in polar coordinates, that is, $\gamma(s) = (\rho(s) \cos \phi(s), \rho(s) \sin \phi(s))$, then, the arclength constraint and the shrinker equation (1.2) become

$$\rho_s^2 + \rho^2 \phi_s^2 = 1, \quad (2.4)$$

$$\rho^2 \phi_s + \rho \rho_{ss} \phi_s - 2\rho_s^2 \phi_s - \rho^2 \phi_s^3 - \rho \rho_s \phi_{ss} = 0,$$

moreover,

$$\cos(\text{angle between } \gamma \text{ and } \gamma_s) = \frac{\gamma \cdot \gamma_s}{|\gamma| |\gamma_s|} = \rho_s. \quad (2.5)$$

Notice that shrinking curves with positive energy have $\phi_s > 0$ everywhere, indeed, either ϕ_s is always different by zero or the curve is a segment of a straight line for the origin of \mathbb{R}^2 .

The curvature and the Energy $\mathcal{E} = ke^{-|\gamma|^2/2}$ are given by

$$k = \rho^2 \phi_s, \quad \mathcal{E} = \rho^2 \phi_s e^{-\frac{1}{2}\rho^2} \quad (2.6)$$

and, when the energy is positive, it will be useful to consider also the quantity $\mathcal{F} := -\log(\mathcal{E})$, that is,

$$\mathcal{F} = -\log(\mathcal{E}) = \frac{1}{2}\rho^2 - \log(\rho^2 \phi_s). \quad (2.7)$$

Since $0 < \rho \phi_s \leq 1$, by equation (2.4), one has

$$\mathcal{F} \geq \frac{1}{2}\rho^2 - \log(\rho) \geq \frac{1}{2}.$$

Let us assume that γ is a shrinking curve with $k > 0$ (the assumption on the sign of k is not restrictive, up to a change of orientation of the curve). Then, by the definition of the Energy (2.1), it is immediate to see that the points where k attains its maximum (resp. minimum) coincide with the points where ρ attains its maximum (resp. minimum). Thus, at any extremal point of k there hold $k_\theta = 0$, $\rho_s = 0$ and also $\rho \phi_s = 1$, by equation (2.4), hence, by equation (2.6), we have $k = \rho$. Then, computing E and \mathcal{F} at such point (clearly, $k_\theta = 0$), we get

$$E = k_{\min}^2 - 2 \log k_{\min} = k_{\max}^2 - 2 \log k_{\max} \quad \text{and} \quad \mathcal{F} = k_{\min}^2/2 - \log k_{\min} = k_{\max}^2/2 - \log k_{\max},$$

that is, $E = 2\mathcal{F} = \log\left(\frac{1}{\mathcal{E}^2}\right)$.

Since the Energy and the quantity \mathcal{F} are constant, this relation must hold along all the curve γ and $\mathcal{F} = \rho_{\min}^2/2 - \log \rho_{\min} = \rho_{\max}^2/2 - \log \rho_{\max}$.

Since the function $\mu(t) = t^2/2 - \log t$ is strictly convex with a minimum value $1/2$ at $t = 1$, to each value of $\mathcal{F} \geq \frac{1}{2}$, there correspond two values $\rho_{\min}(\mathcal{F})$ and $\rho_{\max}(\mathcal{F})$ which are the admissible (interior) minimum and maximum of ρ on γ , with $\rho_{\min}(\mathcal{F}) < 1 < \rho_{\max}(\mathcal{F})$ if $\mathcal{F} > \frac{1}{2}$. It follows easily that $\rho_{\max} : (1/2, +\infty) \rightarrow (1, +\infty)$ is an increasing function and $\rho_{\min} : (1/2, +\infty) \rightarrow (0, 1)$ is a decreasing function. Viceversa, the quantity \mathcal{F} can be seen as a decreasing function of $\rho_{\min} \in (0, 1]$ and an increasing function of $\rho_{\max} \in [1, +\infty)$.

Let $s_{\min}, s_{\max} \in \mathbb{R}$ with $s_{\min} < s_{\max}$ be two consecutive (interior) extremal points of ρ (hence, also of k) such that $\rho(s_{\min}) = \rho_{\min}(\mathcal{F})$, $\rho(s_{\max}) = \rho_{\max}(\mathcal{F})$. Since at the interior extremal points of ρ the vectors γ, γ_s must be orthogonal, it follows that the quantity considered in formula (2.3) satisfies

$$\Delta\theta = \int_{s_{\min}}^{s_{\max}} \phi_s(s) ds := \mathcal{I}(\mathcal{F}), \quad (2.8)$$

that is, the integral $\mathcal{I}(\mathcal{F})$ is the variation of the angle ϕ on the shortest arc such that ρ passes from ρ_{\min} to ρ_{\max} .

Then, by the above discussion, $\mathcal{I}(\mathcal{F}) = I(E) = I(2\mathcal{F})$ and we can rephrase Proposition 2.1 in terms of the integral $\mathcal{I}(\mathcal{F})$ as follows.

Proposition 2.2. *The function $\mathcal{I} : (1/2, +\infty) \rightarrow \mathbb{R}$ satisfies*

- (1) $\lim_{\mathcal{F} \rightarrow (1/2)^+} \mathcal{I}(\mathcal{F}) = \frac{\pi}{\sqrt{2}}$,
- (2) $\lim_{\mathcal{F} \rightarrow +\infty} \mathcal{I}(\mathcal{F}) = \frac{\pi}{2}$,
- (3) $\mathcal{I}(\mathcal{F})$ is monotone decreasing.

As a consequence $\mathcal{I}(\mathcal{F}) > \frac{\pi}{2}$ for all $\mathcal{F} > \frac{1}{2}$.

3. THE PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is based on the following lemma whose proof is postponed to Section 4.

Lemma 3.1. *Let γ be a shrinking curve, parametrized counterclockwise by arclength, with positive curvature and let (s_0, s_1) be an interval where $s \mapsto \rho(s)$ is increasing. If $\rho_s(s_0) \geq \frac{1}{2}$, namely, if the angle formed by the vectors $\gamma(s_0)$ and $\gamma_s(s_0)$ is $\leq \frac{\pi}{3}$, then*

$$\int_{s_0}^{s_1} \phi_s(s) ds < \frac{\pi}{2}. \quad (3.1)$$

Similarly, if $s \mapsto \rho(s)$ is decreasing on (s_0, s_1) and $\rho_s(s_1) \leq -\frac{1}{2}$, namely the angle formed by the vectors $\gamma(s_1)$ and $\gamma_s(s_1)$ is $\geq \frac{2\pi}{3}$, then the same conclusion holds.

Remark 3.2. Proving estimate (3.1) is equivalent to show that

$$\int_{s_0}^{s_1} \theta_s(s) ds < \frac{2\pi}{3}, \quad (3.2)$$

where $\theta(s)$ is the angle formed by $e_1 = (1, 0)$ and the normal vector $\nu(s)$. Indeed, clearly

$$\int_{s_0}^{s_1} \theta_s(s) ds \leq \int_{\sigma_0}^{\sigma_1} \theta_s(s) ds, \quad \int_{s_0}^{s_1} \phi_s(s) ds \leq \int_{\sigma_0}^{\sigma_1} \phi_s(s) ds,$$

where $k(\sigma_1) = k_{\max}$, and σ_0 is the maximum $\sigma \leq s_0$, assuming it exists, such that the angle formed by the vectors $\gamma(\sigma)$ and $\gamma_s(\sigma)$ equals $\frac{\pi}{3}$ and the map $s \mapsto \rho(s)$ is increasing on (σ, σ_1) . Then one observes (by elementary angle geometry) that

$$\int_{\sigma_0}^{\sigma_1} \theta_s(s) ds = \int_{\sigma_0}^{\sigma_1} \phi_s(s) ds + \frac{\pi}{6}.$$

The integral in (3.2) can be expressed as before

$$\int_{s_0}^{s_1} \theta_s(s) ds = \int_{k(s_0)}^{k(s_1)} \frac{dk}{\sqrt{E - k^2 + \log k^2}},$$

hence it is bounded by $I(E)$, defined in formula (2.3) (because, in general, $k_{\min} \leq k(s_0) \leq k(s_1) \leq k_{\max}$). We know that $I(E) < \frac{\pi}{\sqrt{2}}$, but being $\frac{2\pi}{3} < \frac{\pi}{\sqrt{2}}$, estimate (3.2) is not a direct consequence of Proposition 2.1.

Even if such integral is well studied, we found it easier to prove estimate (3.1) than to show that

$$\int_{s_0}^{s_1} \theta_s(s) ds = \int_{k(s_0)}^{k(s_1)} \frac{dk}{\sqrt{E - k^2 + \log k^2}} < \frac{2\pi}{3}$$

and this is the reason for our introduction and computation in polar coordinates (ρ, ϕ) .

We assume now that a Θ -shrinker exists, described by three embedded shrinking curves $\gamma_i : [\underline{s}_i, \bar{s}_i] \rightarrow \mathbb{R}^2$, parametrized by arclength, expressed in polar coordinates by $\gamma_i = (\rho_i \cos(\phi_i), \rho_i \sin(\phi_i))$, for $i \in \{1, 2, 3\}$. The two triple junctions will be denoted with A, B and the three curves intersect each other only at A and B (which are their endpoints) forming angles of 120 degrees. Since the shrinker equation (1.1) is invariant by rotation, we can assume that the segment \overline{AB} is contained in the straight line $\{(x, q) : x \in \mathbb{R}\}$ with $q \geq 0$ and we let $A = (x_A, q)$, $B = (x_B, q)$ with $x_A < x_B$.

We begin with some preliminary elementary lemmas.

Notation. For the sake of simplicity, in all this section we will denote the arclength derivative $\frac{d}{ds}$ with $'$.

Lemma 3.3. For all $i \in \{1, 2, 3\}$, the curve γ_i is either a straight line or such that

$$\left| \int_{\underline{s}_i}^{\bar{s}_i} \phi'_i(s) ds \right| < 2\pi.$$

Proof. Without loss of generality, assume that all $\gamma_1, \gamma_2, \gamma_3$ start at B and end at A , namely $\gamma_i(\underline{s}_i) = B$, $\gamma_i(\bar{s}_i) = A$, for $i \in \{1, 2, 3\}$.

Assume, by contradiction, that γ_1 is a curve with positive energy and curvature such that

$$\int_{\underline{s}_1}^{\bar{s}_1} \phi'_1(s) ds \geq 2\pi.$$

Then there exist $\sigma_1, \tau_1 \in S_1$ such that

$$\int_{\underline{s}_1}^{\sigma_1} \phi'_1(s) ds = 2\pi, \quad \int_{\tau_1}^{\bar{s}_1} \phi'_1(s) ds = 2\pi.$$

Since γ_1 does not intersect itself, one has $(\rho_1(\sigma_1) - \rho_1(\underline{s}_1))(\rho_1(\bar{s}_1) - \rho_1(\tau_1)) > 0$. Assume, without loss of generality, that

$$\rho_1(\sigma_1) < \rho_1(\underline{s}_1), \quad \rho_1(\tau_1) > \rho_1(\bar{s}_1).$$

Now consider the triple junction at the point B , the straight line r passing through B and the origin, and let H_1 and H_2 be the open half-planes in which r divides \mathbb{R}^2 , where H_1 is the one containing $\gamma'_1(\underline{s}_1)$. Since the three curves $\gamma_1, \gamma_2, \gamma_3$ form angles of $\frac{2\pi}{3}$ at B , at least one among $\gamma'_2(\underline{s}_2)$ and $\gamma'_3(\underline{s}_3)$ belongs to H_2 . Without loss of generality, let $\gamma'_2(\underline{s}_2) \in H_2$. Since ϕ'_2 never vanishes, the curve γ_2 cannot reach the endpoint A without crossing the curve γ_1 at some interior point, which is a contradiction. \square

Lemma 3.4. Let $S = [\underline{s}, \bar{s}]$ and $\gamma : S \rightarrow \mathbb{R}^2$ be a shrinking curve parametrized by arclength, expressed in polar coordinates by $\gamma = (\rho \cos(\phi), \rho \sin(\phi))$. Assume that $\phi'(s) > 0$ in S and

$$0 < \Delta \leq \pi, \quad \text{where } \Delta := \int_{\underline{s}}^{\bar{s}} \phi'(s) ds.$$

Let L be the straight line passing through the two points $\gamma(\underline{s}), \gamma(\bar{s})$ and H_1 and H_2 be the two closed half-planes in which L divides the plane \mathbb{R}^2 . Then the arc $\gamma(S)$ is entirely contained in H_1 or H_2 .

Moreover, if $\Delta < \pi$ and $\gamma(S) \subset H_1$, then the origin of \mathbb{R}^2 belongs to the interior of H_2 .

Proof. By the assumption $\phi' > 0$, we have $k > 0$ and the arc $\gamma(S)$ is contained in the cone $\mathcal{C} := \{\phi(\underline{s}) \leq \phi \leq \phi(\bar{s})\}$, which is convex by the assumption $0 < \Delta \leq \pi$. Since the curvature is positive, the closed set \mathcal{T} delimited by the arc $\gamma(S)$ and by the two line segments joining the origin with $\gamma(\underline{s})$ and $\gamma(\bar{s})$ is a convex subset of \mathbb{R}^2 , hence, the line segment joining $\gamma(\underline{s})$ with $\gamma(\bar{s})$ is contained in \mathcal{T} , which implies the thesis. \square

Coming back to our Θ -shrinker, because of its topological structure, one of the curves is contained in the region delimited by the other two, moreover the curvature of both these two “external” curves is always non-zero, otherwise any such curve is a segment of a straight line passing for the origin, then the 120 degrees condition at its endpoints would imply that it must be contained in the region bounded by the other two curves, hence it could not be “external”. Notice that, on the contrary, the “inner” curve could actually be a segment for the origin.

We call γ_2 the “inner” curve and, recalling that the origin of \mathbb{R}^2 is not over the straight line through the two triple junctions A and B , parametrizing counterclockwise the three curves, that is $\phi'_i > 0$ (in the case that the “inner” curve γ_2 is not a segment), we call γ_1 the “external” curve which starts at B . By Lemma 3.3, γ_1 reaches the point A after ϕ_1 changes of an angle $\Delta = \int_{\underline{s}_1}^{\bar{s}_1} \phi'_1(s) ds < 2\pi$ equal to the angle

\widehat{BOA} , which is smaller or equal than π . Hence, by Lemma 3.4, all such curve γ_1 stays over the straight line passing for the two triple junctions A and B .

We call γ_3 the other extremal curve, hence since $\phi_1, \phi_3 > 0$, we have

$$\gamma_1(\underline{s}_1) = \gamma_3(\bar{s}_3) = B, \quad \gamma_1(\bar{s}_1) = \gamma_3(\underline{s}_3) = A.$$

Because of the shrinker equation (1.2), all the three curves are convex with respect to the origin. This implies that the origin is contained in the interior of the bounded area A_{13} enclosed by γ_1 and γ_3 (if the origin belongs to γ_1 or γ_3 such curve is a segment and cannot be “external”, as we said before), which also contains $\gamma_2 \subset A_{13}$. We let A_{12} be the region enclosed by the curves γ_1 and γ_2 and we split the analysis into two cases.

Case 1. The origin does not belong to the interior of A_{12} .

Since the curve γ_2 is convex with respect to the origin, by the same argument used above for γ_1 , it is contained in the upper half-plane determined by the straight line for the points A and B .

By the 120 degrees condition it follows that the angle β at B formed by the vector $(1, 0)$ and γ'_1 is at most $\frac{\pi}{3}$. Similarly, also the angle α at A formed by the vector $(1, 0)$ and γ'_1 is at most $\frac{\pi}{3}$. By the convexity of the region delimited by γ_2 and γ_3 containing the origin and again the 120 degrees condition at B , it is then easy to see that the angle at B formed by the vectors γ_1 and γ'_1 is less or equal than $\frac{\pi}{3}$ and analogously, the angle at A formed by γ_1 and γ'_1 is greater or equal than $\frac{2\pi}{3}$.

Hence, by equality (2.5), it follows

$$\rho'_1(\underline{s}_1) \geq \frac{1}{2} > 0, \quad \rho'_1(\bar{s}_1) \leq -\frac{1}{2} < 0.$$

As a consequence, there is a point of maximum radius $s_1^* \in (\underline{s}_1, \bar{s}_1)$ such that $\rho_1(s_1^*) \geq \rho_1(s)$ for all $s \in (\underline{s}_1, \bar{s}_1)$.

The vector $\gamma_1(s_1^*)$ forms an angle $\sigma \geq \frac{\pi}{2}$ with $(1, 0)$ or $(-1, 0)$. Assume that the angle between $\gamma_1(s_1^*)$ and $(1, 0)$ is greater or equal than $\frac{\pi}{2}$ (the other case is analogous, switching A and B). We extend the curve γ_1 (still parametrized by arclength) “before” the point B till it intersects the x -axis at some $\tilde{s}_1 \leq \underline{s}_1$ (this must happen because $\phi_1(s) > 0$ everywhere also on the extended curve) and we consider the (non-reabeled) curve γ_1 defined in the interval $L_1 = [\tilde{s}_1, s_1^*]$. Calling β_0 the angle formed by the vectors $\gamma'_1(\tilde{s}_1)$ and $(1, 0)$, by convexity and the fact that the angle β at B formed by the vector $(1, 0)$ and γ'_1 is at most $\frac{\pi}{3}$, we have that $\beta_0 \leq \beta \leq \frac{\pi}{3}$. Hence, by equality (2.5), we have $\rho'_1(\tilde{s}_1) \geq \frac{1}{2} > 0$.

Considering now the function $s \mapsto \rho_1(s)$ on the interval $L_1 = [\tilde{s}_1, s_1^*]$, since $\rho'_1(\tilde{s}_1) > 0$ and s_1^* is a maximum point for ρ_1 , either ρ_1 is increasing on L_1 , or ρ_1 has another maximum and then a minimum in the interior of L_1 (notice that the map ρ_1 cannot be constant on an interval, otherwise γ_1 would be an arc of a circle centered at the origin, which is impossible since ρ_1 is not constant). But we know from formula (2.3) and Proposition 2.1 that the angle ϕ_1 must increase more than $\frac{\pi}{2}$ to go from a minimum to a maximum or viceversa (we can apply such proposition since γ_1 is not an arc of a circle). Since

$$\int_{\tilde{s}_1}^{s_1^*} \phi'_1(s) ds \leq \pi,$$

there cannot be a maximum, then a minimum, then a second maximum in L_1 . It follows that ρ_1 is increasing in such interval.

This, combined with the fact that $\beta_0 \leq \frac{\pi}{3}$ and that the angle σ is at least $\frac{\pi}{2}$, that is, $\int_{\tilde{s}_1}^{s_1^*} \phi'_1(s) ds \geq \frac{\pi}{2}$, is in contradiction with Lemma 3.1. Therefore, this case cannot happen.

Case 2. The origin belongs to the interior of A_{12} .

Being the region A_{12} convex (by the shrinker equation (1.2), since it contains the origin), the curve γ_2 (which is oriented counterclockwise) goes from A to B . The fact that γ'_2 and γ'_3 form angles of $\frac{2\pi}{3}$ at the points A and B implies that:

(i) the angle in A formed by the vectors $\gamma_3(\underline{s}_3)$ and $\gamma'_3(\underline{s}_3)$ and the angle in B formed by the vectors $\gamma_2(\bar{s}_2)$ and $\gamma'_2(\bar{s}_2)$ are both less or equal than $\frac{\pi}{3}$;

(ii) the angle in B formed by the vectors $\gamma_3(\bar{s}_3)$ and $\gamma'_3(\bar{s}_3)$ and the angle in A formed by the vectors $\gamma_2(\underline{s}_2)$ and $\gamma'_2(\underline{s}_2)$ are both greater or equal than $\frac{2\pi}{3}$.

In particular, by equality (2.5), it follows

$$\rho'_2(\underline{s}_2) \leq -\frac{1}{2} < 0, \quad \rho'_2(\bar{s}_2) \geq \frac{1}{2} > 0, \quad \rho'_3(\underline{s}_3) \geq \frac{1}{2} > 0, \quad \rho'_3(\bar{s}_3) \leq -\frac{1}{2} < 0. \quad (3.3)$$

Hence, the function $s \mapsto \rho_3(s)$ has a maximum at some point $s_3^* \in (\underline{s}_3, \bar{s}_3)$, while the function $s \mapsto \rho_2(s)$ has a minimum at some point $s_2^\circ \in (\underline{s}_2, \bar{s}_2)$.

If s_3^* is the only point of maximum of ρ_3 in the interval $[\underline{s}_3, \bar{s}_3]$, then the function ρ_3 is strictly monotone on each of the two subintervals $[\underline{s}_3, s_3^*]$ and $[s_3^*, \bar{s}_3]$, moreover,

$$\int_{\underline{s}_3}^{s_3^*} \phi_3'(s) ds + \int_{s_3^*}^{\bar{s}_3} \phi_3'(s) ds = \int_{\underline{s}_3}^{\bar{s}_3} \phi_3'(s) ds \geq \pi,$$

since the origin is “below” the segment \overline{AB} . Thus, at least one of the two integrals on the left-hand side is greater or equal than $\frac{\pi}{2}$ and, by Lemma 3.1, this is not possible. As a consequence, there must be another point of maximum radius $s_3^{**} \in (\underline{s}_3, \bar{s}_3)$ (notice that the maximum points cannot be an interval, otherwise γ_3 would be an arc of a circle centered at the origin, hence with $\rho_3' = 0$, against relations (3.3)). Hence, between these two points of maximum radius there is a minimum point s_3° . Without loss of generality, we assume that $\underline{s}_3 < s_3^* < s_3^\circ < s_3^{**} < \bar{s}_3$.

We observe that there cannot be a third maximum point for ρ_3 (hence also another minimum point) in the interval $[\underline{s}_3, \bar{s}_3]$ because, by Proposition 2.2, each of the four angles at the origin formed by the segment connecting the origin with two consecutive of the five extremal points for ρ_3 on γ_3 is greater than $\frac{\pi}{2}$ and, by Lemma 3.3, there holds $\int_{\underline{s}_3}^{\bar{s}_3} \phi_3'(s) ds < 2\pi$. Moreover, also the case of two minimum points and two maximum points for ρ_3 in the interval $[\underline{s}_3, \bar{s}_3]$ is not possible, because of the sign of the derivative ρ_3' at the endpoints in relations (3.3). Hence, we conclude that $s_3^*, s_3^\circ, s_3^{**}$ are the only extremal points for ρ_3 in the interval $[\underline{s}_3, \bar{s}_3]$.

Now consider the quantities $\mathcal{F}_2, \mathcal{F}_3$ of the curves γ_2, γ_3 , respectively, given by formula (2.7). By relations (3.3), the curves γ_2 and γ_3 are not the unit circle (they would have ρ_2' or ρ_3' equal to zero everywhere), therefore $\mathcal{F}_2, \mathcal{F}_3 > \frac{1}{2}$. If we draw the line from the origin to $\gamma_3(s_3^\circ)$, this must intersect γ_2 in an intermediate point, implying that the minimal radius of the curve γ_2 is smaller than the minimal radius of the curve γ_3 . By the discussion about the value of the quantity \mathcal{F} in relation with the extremal values of ρ at the end of Section 2, we have $\mathcal{F}_2 > \mathcal{F}_3$. Then, if a maximum of ρ_2 is taken in the interior of γ_2 , it must be larger than the maximal radius of γ_3 (which is taken in the interior of γ_3), which is not possible as γ_2 is contained in the region bounded by γ_3 and the segment \overline{AB} . From this argument we conclude that there are no points of maximal radius in the interior of γ_2 , thus, the only extremal point for ρ_2 in the interval $[\underline{s}_2, \bar{s}_2]$ is the minimum point s_2° .

Defining the angle

$$\alpha := \int_{\underline{s}_2}^{\bar{s}_2} \phi_2'(s) ds = \int_{\underline{s}_3}^{\bar{s}_3} \phi_3'(s) ds,$$

by formula (2.8) and the symmetry of the curve γ_3 with respect to the straight line through the origin and the point $\gamma_3(s_3^\circ)$ of minimum distance, we have

$$\mathcal{I}(\mathcal{F}_3) = \int_{s_3^*}^{s_3^\circ} \phi_3'(s) ds = \int_{s_3^\circ}^{s_3^{**}} \phi_3'(s) ds < \frac{\alpha}{2}$$

while, since γ_2 does not contain any interior point of maximum radius,

$$\mathcal{I}(\mathcal{F}_2) > \max \left\{ \int_{\underline{s}_2}^{s_2^\circ} \phi_2'(s) ds, \int_{s_2^\circ}^{\bar{s}_2} \phi_2'(s) ds \right\} \geq \frac{\alpha}{2}.$$

Thus, $\mathcal{I}(\mathcal{F}_2) > \mathcal{I}(\mathcal{F}_3)$ and $\mathcal{F}_2 > \mathcal{F}_3$, which is in contradiction with the monotonicity of the function \mathcal{I} given by Proposition 2.2. Hence, also this case can be excluded.

Since we excluded both cases, our hypothetical Θ -shrinker cannot exist (under the validity of Lemma 3.1 which we are going to prove in the next section).

4. THE PROOF OF LEMMA 3.1

We prove Lemma 3.1 in the case of ρ increasing, for ρ decreasing the proof is similar and it can be deduced by changing the orientation of the curve.

Let $\gamma = (\rho \cos \phi, \rho \sin \phi)$ be a counterclockwise arclength parametrized shrinking curve, that we suppose defined for every $s \in \mathbb{R}$, and let $[s_0, s_1]$ be an interval where $\rho(s)$ is increasing. As a first step, we

prove a formula for the integral in Lemma 3.1, using ρ as a new integration variable. Let $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}$ be the function

$$\mu(x) := \frac{1}{2}x^2 - \log(x), \quad (4.1)$$

so that relation (2.7) gives

$$\mathcal{F} = \mu(\rho(s)) - \log(\rho(s)\phi_s(s)) \quad \forall s \in \mathbb{R}. \quad (4.2)$$

Let R be the maximum value of ρ along the whole γ , namely $R = \rho_{\max}(\mathcal{F})$ in the notation at the end of Section 2. We consider $R > 1$, namely we exclude the case when γ is the unit circle (for the unit circle Lemma 3.1 trivially holds, as $\rho_s = 0$ everywhere). Since $\rho_s = 0$ when $\rho = R$, by equation (2.4) one has $\rho\phi_s = 1$ at that point, therefore, by relation (4.2), we deduce

$$\mathcal{F} = \mu(R), \quad (4.3)$$

which gives a link between the quantity \mathcal{F} and the maximum distance R of the curve γ . Notice that, for the same reason, $\mathcal{F} = \mu(r)$ where $r = \rho_{\min}(\mathcal{F})$.

By equations (4.2) and (4.3), it follows

$$\rho\phi_s = \exp(\mu(\rho) - \mu(R)),$$

hence,

$$\phi_s = \frac{1}{\rho} e^{\mu(\rho) - \mu(R)},$$

then, since $\rho_s > 0$ in $[s_0, s_1]$, by equation (2.4), we conclude

$$\rho_s = \sqrt{1 - e^{2[\mu(\rho) - \mu(R)]}}. \quad (4.4)$$

Therefore, changing the integration variable as $\rho = \rho(s)$, we get

$$\int_{s_0}^{s_1} \phi_s(s) ds = \int_{\rho(s_0)}^{\rho(s_1)} f(\rho, R) d\rho \quad (4.5)$$

where

$$f(\rho, R) := \frac{e^{\mu(\rho) - \mu(R)}}{\rho\sqrt{1 - e^{2[\mu(\rho) - \mu(R)]}}}. \quad (4.6)$$

The function $f(\rho, R)$ is defined for $R > 1$ and $\rho \in (r, R)$, where $r = \rho_{\min}(\mu(R)) \in (0, 1)$.

Let $s_{\min} < s_{\max}$ be two points such that

$$\rho_s(s_{\min}) = 0, \quad \rho_s(s_{\max}) = 0, \quad \rho_s(s) > 0 \quad \forall s \in (s_{\min}, s_{\max}),$$

so that $\rho(s_{\min}) = r = \rho_{\min}(\mathcal{F})$ and $\rho(s_{\max}) = R = \rho_{\max}(\mathcal{F})$. By equation (2.4), $\rho_s(s)$ is increasing when $\rho(s)\phi_s(s)$ is decreasing; being the quantity \mathcal{F} constant along the curve γ , by equation (4.2), this happens when $\mu(\rho(s))$ is decreasing, namely when $\mu'(\rho(s)) < 0$, that is, for $\rho(s) < 1$ (the function μ is strictly convex with a minimum at $x = 1$). Therefore, ρ_s is increasing on $[s_{\min}, \sigma]$ and decreasing on $[\sigma, s_{\max}]$, where $\sigma \in (s_{\min}, s_{\max})$ is the only point such that $\rho(\sigma) = 1$.

We analyze now the points where $\rho_s \geq \frac{1}{2}$. First we observe that if the quantity \mathcal{F} is too low, then there are no such points. Indeed, since ρ_s has its maximum when $\rho = 1$, namely at $s = \sigma$, by equation (4.4), we have

$$\rho_s(\sigma) = \sqrt{1 - e^{1 - 2\mu(R)}} = \sqrt{1 - e^{1 - 2\mathcal{F}}},$$

as $\mu(\rho(\sigma)) = \mu(1) = 1/2$.

It follows that $\rho_s(\sigma) \geq \frac{1}{2}$ if and only if

$$\mathcal{F} \geq \frac{1}{2} - \log\left(\frac{\sqrt{3}}{2}\right),$$

that is, if and only if $R \geq \bar{R}$, where \bar{R} is the unique real number such that

$$\mu(\bar{R}) = \frac{1}{2} - \log\left(\frac{\sqrt{3}}{2}\right), \quad \bar{R} > 1. \quad (4.7)$$

As a consequence, Lemma 3.1 trivially holds for curves γ such that $R < \bar{R}$ (because the assumption $\rho_s(s_0) \geq \frac{1}{2}$ is simply not satisfied).

Let $R \geq \bar{R}$. Let \hat{s} be the smallest number in the interval (s_{\min}, s_{\max}) such that $\rho_s(\hat{s}) \geq \frac{1}{2}$, namely \hat{s} is the unique number in $(s_{\min}, \sigma]$ such that $\rho_s(\hat{s}) = \frac{1}{2}$, then, clearly

$$\int_{s_0}^{s_1} \phi_s(s) ds \leq \int_{\hat{s}}^{s_{\max}} \phi_s(s) ds. \quad (4.8)$$

Letting $d(R) = \rho(\hat{s})$, by equation (4.4), $d(R)$ is the unique solution of

$$\mu(d(R)) = \mu(R) + \log\left(\frac{\sqrt{3}}{2}\right), \quad 0 < d(R) \leq 1, \quad (4.9)$$

for any $R \geq \bar{R}$. Notice that $d(\bar{R}) = 1$ and $0 < d(R) < 1$ for $R > \bar{R}$. Finally, by equations (4.5) and (4.8), we have

$$\int_{s_0}^{s_1} \phi_s(s) ds \leq \int_{\hat{s}}^{s_{\max}} \phi_s(s) ds = \int_{d(R)}^R f(x, R) dx =: J(R).$$

Lemma 3.1 is then proved once we get the following bound.

Proposition 4.1. *For all $R \geq \bar{R}$ there holds*

$$J(R) < \frac{\pi}{2}.$$

4.1. Proof of Proposition 4.1 – Preliminaries.

It is easy to see by the definition of $d(R)$ in formula (4.9) that $R \mapsto d(R)$ is a decreasing function on $[\bar{R}, +\infty)$ with $d(\bar{R}) = 1$. We let

$$\psi(t) := \frac{t}{\sqrt{1-t^2}}, \quad t \in (0, 1), \quad (4.10)$$

then, the function f , defined by formula (4.6), can be expressed as

$$f(\rho, R) = \frac{e^{\frac{1}{2}(\rho^2 - R^2)} R}{\rho^2 \sqrt{1 - e^{\rho^2 - R^2}} R^2 \rho^{-2}} = \frac{e^{\mu(\rho) - \mu(R)}}{\rho \sqrt{1 - e^{2[\mu(\rho) - \mu(R)]}}} = \frac{1}{\rho} \psi(\exp\{\mu(\rho) - \mu(R)\}),$$

for $\rho \in [d(R), R]$. Indeed, $\mu(\rho) - \mu(R) < 0$ for all $\rho \in [d(R), R]$.

Notice that, even if $f(\rho, R) \rightarrow +\infty$ as $\rho \rightarrow R^-$, the integral $J(R)$ is finite for all $R \geq \bar{R}$, because f diverges like $(R - \rho)^{-1/2}$ as $\rho \rightarrow R^-$.

Lemma 4.2 (Approximation of \bar{R}). *Recalling the defining formula (4.7) for \bar{R} , we have*

$$\sqrt{1 + \log(c_n)} < \bar{R} < \sqrt{1 + \log(d_n)} \quad \forall n = 0, 1, 2, \dots \quad (4.11)$$

where (c_n) is the increasing sequence defined by

$$c_0 = 1, \quad c_{n+1} = \frac{4}{3}(1 + \log(c_n)), \quad n \geq 0 \quad (4.12)$$

and (d_n) is the decreasing sequence defined by

$$d_0 = 3, \quad d_{n+1} = \frac{4}{3}(1 + \log(d_n)), \quad n \geq 0. \quad (4.13)$$

Proof. By definition (4.7), $\bar{R} > 1$ and the first inequality in (4.11) for $n = 0$ holds. Since $\frac{5}{4} > \log(3)$, one directly proves that $\mu(\sqrt{1 + \log(3)}) > \frac{1}{2} - \log\left(\frac{\sqrt{3}}{2}\right) = \mu(\bar{R})$, where μ is defined by formula (4.1). Since μ is increasing on $(1, +\infty)$, it follows that $\bar{R} < \sqrt{1 + \log(3)}$, which is the second inequality in (4.11) for $n = 0$.

Assume that $\sqrt{1 + \log(c_n)} < \bar{R} < \sqrt{1 + \log(d_n)}$ for some $n \geq 0$. Then

$$1 + \log\left(\frac{4}{3}(1 + \log(c_n))\right) < 1 + \log\left(\frac{4}{3}\bar{R}^2\right) < 1 + \log\left(\frac{4}{3}(1 + \log(d_n))\right).$$

By definition of \bar{R} one has $1 + \log\left(\frac{4}{3}\bar{R}^2\right) = \bar{R}^2$, hence, by equations (4.12) and (4.13), we get $1 + \log(c_{n+1}) < \bar{R}^2 < 1 + \log(d_{n+1})$. Thus, formula (4.11) holds for all $n \geq 0$.

For all $n \geq 1$ one has $c_{n+1} > c_n$ if and only if $c_n > c_{n-1}$ and similarly $d_{n+1} < d_n$ if and only if $d_n < d_{n-1}$. Since $c_1 > c_0$ and $d_1 < d_0$, the sequence (c_n) is increasing and (d_n) is decreasing. \square

Lemma 4.3 (Exact rounding for the value of \bar{R}). *There holds $\frac{21}{16} < \frac{7}{5} < \bar{R} < \frac{23}{16}$.*

Proof. Using *interval arithmetic*, we compute exact roundings for c_n, d_n in Lemma 4.2 for $n = 50$, and we get $1.4004566266453120082 < \bar{R} < 1.4004566266453162271$. \square

Lemma 4.4 (Approximation of $d(R)$). Let $R \geq \bar{R}$, recalling the defining formula (4.9) for $d(R)$ and denoting

$$a := -\mu(R) - \log\left(\frac{\sqrt{3}}{2}\right), \quad (4.14)$$

there holds

$$\exp(a_n) < d(R) \leq \exp(b_n), \quad \forall n = 0, 1, 2, \dots \quad (4.15)$$

where (a_n) is the increasing sequence defined by

$$a_0 = a, \quad a_{n+1} = a + \frac{1}{2} \exp(2a_n), \quad n \geq 0 \quad (4.16)$$

and (b_n) is defined by

$$b_0 = a + \frac{1}{2}, \quad b_{n+1} = a + \frac{1}{2} \exp(2b_n), \quad n \geq 0,$$

which is a decreasing sequence if $R > \bar{R}$. It is constant equal to zero if $R = \bar{R}$.

Proof. Since $d(R) > 0$, from formulas (4.9) and (4.14) one has $\log(d(R)) = a + \frac{1}{2}d(R)^2 > a$, hence $d(R) > \exp(a_0)$. By induction, assume that $d(R) > \exp(a_n)$ for some $n \geq 0$. Then $\log(d(R)) = a + \frac{1}{2}d(R)^2 > a + \frac{1}{2} \exp(2a_n) = a_{n+1}$, hence $d(R) > \exp(a_{n+1})$. This proves the first inequality (4.15) for all $n \geq 0$. Since $d(R) \leq 1$, one has $\log(d(R)) = a + \frac{1}{2}d(R)^2 \leq a + \frac{1}{2}$, hence $d(R) \leq \exp(b_0)$. Assume that $d(R) \leq \exp(b_n)$ for some $n \geq 0$. Then $\log(d(R)) = a + \frac{1}{2}d(R)^2 \leq a + \frac{1}{2} \exp(2b_n) = b_{n+1}$, hence $d(R) \leq \exp(b_{n+1})$. The proof of inequalities (4.15) is complete. By the definition (4.16), it follows that $a_{n+1} > a_n$ if and only if $a_n > a_{n-1}$, for all $n \geq 1$. Moreover, $a_1 > a_0$, therefore (a_n) is increasing. Similarly, $b_{n+1} - b_n$ has the same sign of $b_n - b_{n-1}$ for all $n \geq 1$. For $R > \bar{R}$ one has $b_0 < 0$, hence $b_1 < b_0$ and the sequence (b_n) is decreasing. Notice that for $R = \bar{R}$ one has $b_n = b_0 = 0$ for all n and $d(\bar{R}) = 1 = \exp(b_n)$. \square

To approximate the integral $J(R)$ from above, we will use the convexity of the function $f(\cdot, R)$.

Lemma 4.5. For all $R \geq \bar{R}$, the function $\rho \mapsto f(\rho, R)$ is strictly convex on $[d(R), R)$.

Proof. By definition, $f(\rho, R) = \frac{1}{\rho} \psi(\mathcal{G})$, where $\mathcal{G} := \mathcal{G}(\rho, R) = \exp(\mu(\rho) - \mu(R))$ and ψ, μ are defined in formulas (4.1) and (4.10). We have $\mu'(\rho) = \rho - \rho^{-1}$, $\psi'(x) = (1 - x^2)^{-3/2}$, and $\partial_\rho \mathcal{G} = (\rho - \rho^{-1})\mathcal{G}$. Thus,

$$\partial_\rho f(\rho, R) = \mathcal{G}(1 - \mathcal{G}^2)^{-\frac{3}{2}} [1 + (\mathcal{G}^2 - 2)\rho^{-2}], \quad \partial_{\rho\rho}^2 f(\rho, R) = \frac{\mathcal{G}}{(1 - \mathcal{G}^2)^{\frac{5}{2}} \rho^3} h$$

where

$$h := \rho^4(1 + 2\eta) - 3\rho^2(1 + \eta) + (2\eta^2 - 5\eta + 6), \quad \eta := \mathcal{G}^2.$$

Then,

$$h = \left(\rho^2 \sqrt{1 + 2\eta} - \frac{3(1 + \eta)}{2\sqrt{1 + 2\eta}} \right)^2 + \frac{1 - \eta}{4(1 + 2\eta)} (15 + 25\eta - 16\eta^2),$$

and $0 < \eta < 1$ for all $R \in [d(R), R)$ because $\mu(\rho) - \mu(R) < 0$. Hence, $1 - \eta > 0$ and also $15 + 25\eta - 16\eta^2 > 0$, for all $\eta \in (0, 1)$. Therefore, $h > 0$ and $\partial_{\rho\rho}^2 f(\rho, R) > 0$, for all $\rho \in [d(R), R)$, $R \geq \bar{R}$. \square

Lemma 4.6. For all $R > \bar{R}$, all $\rho \in [d(R), R)$, there holds $\partial_R f(\rho, R) < 0$, that is, the function $R \mapsto f(\rho, R)$ is decreasing.

Proof. With the same notation as in the proof of Lemma 4.5, we have

$$\partial_R f(\rho, R) = -\rho^{-1} \mathcal{G}(1 - \mathcal{G}^2)^{-3/2} (R - R^{-1}) < 0.$$

\square

4.2. Proof of Proposition 4.1 for “small” R .

Lemma 4.7. *There holds $J(R) < \frac{3}{2}$ for all $R \in [\bar{R}, \frac{23}{16}]$.*

Proof. Recalling the bound in Lemma 4.3, we split

$$J(R) = \int_{d(R)}^1 f(\rho, R) d\rho + \int_1^{\rho_2} f(\rho, R) d\rho + \int_{\rho_2}^R f(\rho, R) d\rho, \quad \rho_2 := \frac{21}{16}.$$

Since $f(\cdot, R)$ is convex (Lemma 4.5), the integral over the region $[d(R), 1]$ is bounded by the area of the trapezoid

$$\int_{d(R)}^1 f(\rho, R) d\rho \leq \frac{1-d(R)}{2} (f(d(R), R) + f(1, R)).$$

Now $f(d(R), R) = \sqrt{3}/d(R)$, while, recalling that $d(\bar{R}) = 1$, we have

$$\begin{aligned} f(1, R) &= \psi(\exp\{\mu(1) - \mu(R)\}) \leq \psi(\exp\{\mu(1) - \mu(\bar{R})\}) \\ &= \psi(\exp\{\mu(d(\bar{R})) - \mu(\bar{R})\}) = \psi\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}. \end{aligned}$$

Hence, using that $d(R) \geq d(23/16)$, we obtain

$$\int_{d(R)}^1 f(\rho, R) d\rho \leq \frac{1-d(R)}{2} \left(\sqrt{3} + \frac{\sqrt{3}}{d(R)} \right) \leq \frac{1-d(23/16)}{2} \left(\sqrt{3} + \frac{\sqrt{3}}{d(23/16)} \right).$$

The integral over the region $[1, \rho_2]$ is also bounded by the area of the corresponding trapezoid, namely,

$$\int_1^{\rho_2} f(\rho, R) d\rho < \frac{\rho_2 - 1}{2} \left(\sqrt{3} + f(\rho_2, R) \right). \quad (4.17)$$

The integral over the region $[\rho_2, R]$ is bounded by using the change of variable

$$e^{\mu(\rho) - \mu(R)} = \xi, \quad \frac{(\rho^2 - 1)e^{\mu(\rho) - \mu(R)}}{\rho} d\rho = d\xi, \quad (4.18)$$

which gives

$$\begin{aligned} \int_{\rho_2}^R f(\rho, R) d\rho &= \int_{\rho_2}^R \frac{1}{\rho^2 - 1} \frac{1}{\sqrt{1 - e^{2[\mu(\rho) - \mu(R)]}}} \frac{(\rho^2 - 1)e^{\mu(\rho) - \mu(R)}}{\rho} d\rho \\ &\leq \frac{1}{\rho_2^2 - 1} \int_{\rho_2}^R \frac{1}{\sqrt{1 - e^{2[\mu(\rho) - \mu(R)]}}} \frac{(\rho^2 - 1)e^{\mu(\rho) - \mu(R)}}{\rho} d\rho \\ &= \frac{1}{\rho_2^2 - 1} \int_{\xi(\rho_2)}^{\xi(R)} \frac{1}{\sqrt{1 - \xi^2}} d\xi \\ &= \frac{1}{\rho_2^2 - 1} \left(\frac{\pi}{2} - \arcsin(e^{\mu(\rho_2) - \mu(R)}) \right). \end{aligned} \quad (4.19)$$

Using Lemmas 4.3 and 4.6, we have $f(\rho_2, R) < f(\rho_2, 7/5)$ and $\mu(R) \leq \mu(23/16)$, and we insert these bounds in (4.17) and (4.19). Since $\rho_2 = 21/16$, we get

$$\begin{aligned} J(R) &\leq \frac{1-d(23/16)}{2} \left(\sqrt{3} + \frac{\sqrt{3}}{d(23/16)} \right) + \frac{5}{32} \left(\sqrt{3} + f\left(\frac{21}{16}, \frac{7}{5}\right) \right) \\ &\quad + \frac{256}{185} \left(\frac{\pi}{2} - \arcsin(e^{\mu(21/16) - \mu(23/16)}) \right). \end{aligned} \quad (4.20)$$

Using Lemma 4.4 and *interval arithmetic* for correct roundings, we obtain that the right-hand side of (4.20) is less than 1.4712. \square

4.3. Proof of Proposition 4.1 for “large” R .

Lemma 4.8. *There holds $J(R) < \frac{3}{2}$ for all $R \in [4, +\infty)$.*

Proof. Let $R \geq 4$, we consider some ρ_1, ρ_2 such that $d(R) < \rho_1 < 1 < \rho_2 < R$ that we will choose later and we split the integral $J(R)$ of the four intervals.

(1) The integral over $[d(R), \rho_1]$.

We use the change of variable (4.18). Since μ is decreasing on $(0, 1)$, we get

$$\int_{d(R)}^{\rho_1} f(\rho, R) d\rho \leq \frac{1}{1 - \rho_1^2} \arcsin(\xi) \Big|_{\xi(\rho_1)}^{\xi(d(R))} = \frac{1}{1 - \rho_1^2} \left(\frac{\pi}{3} - \arcsin(\xi(\rho_1)) \right) < \frac{1}{1 - \rho_1^2} \frac{\pi}{3}. \quad (4.21)$$

(2) The integral over $[\rho_1, 1]$.

Since μ is decreasing on $(0, 1)$, we have

$$\begin{aligned} \int_{\rho_1}^1 f(\rho, R) d\rho &= \int_{\rho_1}^1 \frac{1}{\rho} \psi(e^{\mu(\rho) - \mu(R)}) \\ &\leq \psi(e^{\mu(\rho_1) - \mu(R)}) \int_{\rho_1}^1 \frac{1}{\rho} d\rho = \log\left(\frac{1}{\rho_1}\right) \psi(e^{\mu(\rho_1) - \mu(R)}). \end{aligned} \quad (4.22)$$

(3) The integral over $[1, \rho_2]$.

Since μ is increasing on $(1, +\infty)$, we have

$$\begin{aligned} \int_1^{\rho_2} f(\rho, R) d\rho &= \int_1^{\rho_2} \frac{1}{\rho} \psi(e^{\mu(\rho) - \mu(R)}) \\ &\leq \psi(e^{\mu(\rho_2) - \mu(R)}) \int_1^{\rho_2} \frac{1}{\rho} d\rho = \log(\rho_2) \psi(e^{\mu(\rho_2) - \mu(R)}). \end{aligned} \quad (4.23)$$

(4) The integral over $[\rho_2, R]$.

Using the general inequality

$$\psi(e^x) = \frac{e^x}{\sqrt{1 - e^{2x}}} < \frac{1}{\sqrt{2|x|}} \quad \forall x < 0$$

we get

$$\psi(e^{\mu(\rho) - \mu(R)}) < \frac{1}{\sqrt{R^2 - \rho^2 - \log(R^2 \rho^{-2})}}$$

and, since $\log(1 + x) < x$ for all $x > -1$, we deduce that

$$f(\rho, R) < \frac{1}{\sqrt{(R^2 - \rho^2)(\rho^2 - 1)}}.$$

Hence,

$$\int_{\rho_2}^R f(\rho, R) d\rho < \frac{1}{\sqrt{(R + \rho_2)(\rho_2^2 - 1)}} \int_{\rho_2}^R \frac{1}{\sqrt{R - \rho}} d\rho = \frac{2\sqrt{R - \rho_2}}{\sqrt{(R + \rho_2)(\rho_2^2 - 1)}}. \quad (4.24)$$

The sum of the integrals (4.21), (4.22), (4.23) and (4.24), with $\rho_2 = R - 1$ and some ρ_1 , gives

$$J(R) \leq Q_1(\rho_1) + Q_2(\rho_1, R) + Q_3(R) + Q_4(R) =: Q(\rho_1, R) \quad (4.25)$$

where

$$Q_1(\rho_1) := \frac{1}{1 - \rho_1^2} \frac{\pi}{3}, \quad Q_2(\rho_1, R) := \log\left(\frac{1}{\rho_1}\right) \psi(e^{\mu(\rho_1) - \mu(R)}),$$

$$Q_3(R) := \log(R - 1) \psi(e^{\mu(R-1) - \mu(R)}), \quad Q_4(R) := \frac{2}{\sqrt{(2R - 1)(R^2 - 2R)}}.$$

For any ρ_1 , the value $Q_1(\rho_1)$ does not depend on R and $Q_2(\rho_1, R)$, $Q_4(R)$ are decreasing functions of R on $[4, +\infty)$. Computing the derivative of $Q_3(R)$, we see that Q_3 is decreasing if

$$R - e^{1-2R} \frac{R^3}{(R-1)^2} - (R^2 - R + 1) \log(R - 1) < 0.$$

Since

$$R - (R^2 - R) \log(R - 1) = R[1 - (R - 1) \log(R - 1)] < 0$$

for $R \geq 4$, we deduce that also Q_3 is decreasing on $[4, +\infty)$. Hence, the function $R \mapsto Q(\rho_1, R)$ defined in formula (4.25) is decreasing on $[4, +\infty)$, therefore,

$$J(R) \leq Q(\rho_1, R) \leq Q(\rho_1, 4) \quad \forall R \geq 4.$$

We fix $\rho_1 = 1/8$ and we note that $d(R) \leq d(4) < 1/500 < \rho_1 = 1/8$ for all $R \geq 4$ (Lemma 4.4 and exact rounding with *interval arithmetic* give $d(4) < 0.00155$). We compute $Q(1/8, 4) < 1.4$, thus $J(R) < 1.4$, for $R \geq 4$. \square

4.4. Proof of Proposition 4.1 for “intermediate” R .

Lemma 4.9. *There holds $J(R) < 1.52$ for all $R \in [\frac{23}{16}, 4]$.*

Proof. We split the interval $[\frac{23}{16}, 4]$ into M subintervals of length δ_R : we fix

$$\delta_R := 2^{-10}, \quad \mathcal{R}_n := \left[\frac{23}{16} + (n-1)\delta_R, \frac{23}{16} + n\delta_R \right], \quad n = 1, 2, \dots, M, \quad (4.26)$$

where $M := (4 - \frac{23}{16})\delta_R^{-1} = 2624$. Note that all the numbers $\frac{23}{16} + n\delta_R$ with $n \in \mathbb{N} \cap [1, M]$ are represented exactly by any standard computer. Now for each $n = 1, \dots, M$ we estimate the set $\{J(R) : R \in \mathcal{R}_n\}$.

Let \mathcal{R} be any of the intervals \mathcal{R}_n and call R_{min}, R_{max} its extremal points. Let α, β , with $\alpha < \beta$, be two rational numbers that are represented exactly by the computer, such that $d(R) \in [\alpha, \beta]$, for all $R \in \mathcal{R}$. To estimate $J(R)$ for all $R \in \mathcal{R}$, we split the integration interval $[d(R), R]$ into many subintervals.

First, let $\delta_0 = 2^{-m_0}$, $m_0 \in \mathbb{N}$, be such that

$$\beta + \delta_0 \leq 1, \quad (4.27)$$

so that $d(R) + \delta_0 \leq 1$ for all $R \in \mathcal{R}$. Since $f(\rho, R)$ is convex in ρ (Lemma 4.5), the integral over $[d(R), d(R) + \delta_0]$ is bounded by the area of the trapezoid,

$$\int_{d(R)}^{d(R)+\delta_0} f(\rho, R) d\rho \leq \frac{\delta_0}{2} \left(\frac{\sqrt{3}}{d(R)} + f(d(R) + \delta_0, R) \right) =: J_0(R).$$

Next, since $\alpha \leq d(R)$, we have

$$\int_{d(R)+\delta_0}^1 f(\rho, R) d\rho \leq \int_{\alpha+\delta_0}^1 f(\rho, R) d\rho$$

and $[\alpha + \delta_0, 1]$ is contained in the domain $[d(R), R]$ of $f(\cdot, R)$ for all $R \in \mathcal{R}$ if

$$\beta \leq \alpha + \delta_0.$$

We define m_0 as the integer part $[-\log_2(\beta - \alpha)]$, so that $\beta - \alpha \leq \delta_0 < 2(\beta - \alpha)$. Inequality (4.27) holds if α, β satisfy

$$3\beta - 2\alpha \leq 1.$$

We split the integration interval $[\alpha + \delta_0, 1]$ into N_1 subintervals of length $\delta_1 = 2^{-10}$ and, possibly, one smaller interval. Let N_1 be the integer part of $(1 - \alpha - \delta_0)\delta_1^{-1}$ and define

$$x_k := (\alpha + \delta_0) + (k-1)\delta_1, \quad k = 1, \dots, N_1 + 1.$$

Thus, $x_1 = \alpha + \delta_0$ and $1 - \delta_1 < x_{N_1+1} \leq 1$. Note that the numbers x_k are exactly representable by the computer. Since $f(\cdot, R)$ is convex, one has

$$\int_{\alpha+\delta_0}^1 f(\rho, R) d\rho \leq J_1(R) + \int_{x_{N_1+1}}^1 f(\rho, R) d\rho \quad (4.28)$$

where

$$J_1(R) := \frac{\delta_1}{2} \left(f(x_1, R) + f(x_{N_1+1}, R) \right) + \delta_1 \sum_{k=2}^{N_1} f(x_k, R).$$

Next, let $\delta_4 > 0$ be such that $R_{min} - \delta_4 \geq 1$. Bound (4.24) gives

$$\int_{R-\delta_4}^R f(\rho, R) d\rho < \frac{2\sqrt{\delta_4}}{\sqrt{(2R - \delta_4)[(R - \delta_4)^2 - 1]}} =: J_4(R).$$

Since $R \leq R_{max}$,

$$\int_1^{R-\delta_4} f(\rho, R) d\rho \leq \int_1^{R_{max}-\delta_4} f(\rho, R) d\rho,$$

and $[1, R_{max} - \delta_4]$ is contained in the domain $[d(R), R]$ of $f(\cdot, R)$ for all $R \in \mathcal{R}$ if

$$\delta_4 > R_{max} - R_{min} = \delta_R.$$

Since $\delta_R = 2^{-10}$ and $R_{min} \geq \frac{23}{16}$, we can fix $\delta_4 = 2^{-6}$. We split the integration interval $[1, R_{max} - \delta_4]$ into N_3 subintervals of length $\delta_3 = 2^{-6}$ and, possibly, one smaller interval. Let N_3 be the integer part of $(R_{max} - \delta_4 - 1)\delta_3^{-1}$ and define

$$z_k := (R_{max} - \delta_4) - (k-1)\delta_3, \quad k = 1, \dots, N_3 + 1.$$

Thus $z_1 = R_{max} - \delta_4$ and $1 \leq z_{N_3+1} < 1 + \delta_3$. Note that the numbers z_k are exactly representable on the computer. Since $f(\cdot, R)$ is convex,

$$\int_1^{R_{max} - \delta_4} f(\rho, R) d\rho \leq J_3(R) + \int_1^{z_{N_3+1}} f(\rho, R) d\rho, \quad (4.29)$$

where

$$J_3(R) := \frac{\delta_3}{2} \left(f(z_1, R) + f(z_{N_3+1}, R) \right) + \delta_3 \sum_{k=2}^{N_3} f(z_k, R).$$

Regarding the two remaining integrals (4.28) and (4.29), one has

$$\int_{x_{N_1+1}}^1 f(\rho, R) d\rho + \int_1^{z_{N_3+1}} f(\rho, R) d\rho = \int_{x_{N_1+1}}^{z_{N_3+1}} f(\rho, R) d\rho \leq J_2(R)$$

where

$$J_2(R) := \frac{z_{N_3+1} - x_{N_1+1}}{2} \left(f(x_{N_1+1}, R) + f(z_{N_3+1}, R) \right).$$

Thus,

$$J(R) \leq \sum_{i=0}^4 J_i(R). \quad (4.30)$$

We use *interval arithmetic* to find a correct rounding for each term $J_i(R)$, uniformly in $R \in \mathcal{R}$ and we repeat the procedure for all intervals in formula (4.26). We get $J(R) \leq 1.51734 < 1.52$ for all $R \in [\frac{23}{16}, 4]$. The maximum value of the upper bound for the right-hand side of inequality (4.30) is obtained on the interval \mathcal{R}_n with $n = 490$, corresponding to R close to 1.915 (but there is no reason for which the true function $J(R)$ and the upper bound we have computed should have a maximum at the same point). Along the procedure the exponent m_0 of δ_0 assumes integer values between 5 and 17. The computation takes about 18 minutes on our standard computers. The code we have used is in Appendix A. \square

5. SOME CONSEQUENCES AND OPEN QUESTIONS

The arguments of Section 3 can be used to prove the following proposition, also conjectured in [10, Conjecture 3.22].

Proposition 5.1. *Any region of a regular shrinking network bounded by only two curves must contain the origin in its interior.*

Proof. Such a region cannot be strictly convex, otherwise, by the shrinkers equation the conclusion is immediate. If one of the curves is a segment, by reflecting the region with respect to the straight line containing such segment, which must pass through the origin, one would obtain a Θ -shrinker which is excluded by Theorem 1.1. Hence, we suppose that the closed region is bounded by two curves with the “same convexity” with respect to the origin, counterclockwise parametrized by arclength (hence going from the triple junction B to the triple junction A) and we assume that the origin is outside the region. As before, we separate the analysis in two cases, according to the position of the origin with respect to the straight line containing the segment \overline{AB} . The case when the origin is below or belongs to such line follows as in *Case 1* of the proof of Theorem 1.1 in Section 3. The case when the origin is above the straight line containing the segment \overline{AB} , can be treated with the same (energetic) arguments of *Case 2*. \square

As an immediate consequence, not only a Θ -shrinker cannot exist, but also there are no shrinking networks with more than one region bounded by only two curves.

This is another step in the classification of the whole family of shrinkers. For instance, it is easy to show that every bounded region must have less than six bounding curves and its area is determined by such number.

Several other results in this direction were obtained in [10, 25, 26]. We want to mention the work of Hättenschweiler [10] where it is proposed the very interesting Conjecture 3.26 that there is an upper bound for the possible number of bounded regions of a shrinker. This clearly would imply that the possible topological structures of compact shrinkers are finite.

APPENDIX A. CODES FOR INTERVAL ARITHMETIC COMPUTATIONS

This is the code we have used to compute a correct rounding (in particular, an exact upper bound) for the right-hand side of (4.30).

```

pkg load interval
tic ();
output_precision(6)
delta_R = 2^(-10);
delta_1 = 2^(-10);
delta_3 = 2^(-6);
delta_4 = 2^(-6);
NN = 10; %% Number of iterations for the computation of d(R).
if (delta_4 < delta_R)
    printf("Attention: it is delta_4 < delta_R, whereas it should be >=. \n")
endif

M = (4 - 23/16) / delta_R;
J_0 = infsup(zeros(M,1));
J_1 = infsup(zeros(M,1));
J_2 = infsup(zeros(M,1));
J_3 = infsup(zeros(M,1));
J_4 = infsup(zeros(M,1));
J = infsup(zeros(M,1));
vec_R = infsup(zeros(M,1));
vec_dR = infsup(zeros(M,1));
m_0 = infsup(zeros(M,1));

for n=1:M
    R_left = 23/16 + (n-1) * delta_R;
    R_right = R_left + delta_R;
    R = infsup( R_left , R_right ); % R is an interval temporary variable
    vec_R(n) = R; % the value of R is stored in the vector vec_R at row n

    %% Here we calculate d(R) as a correctly rounding interval:
    b = R^2 / 2 - log(R) + (1/2) * log(infsup(3)) - log(infsup(2));
    a = -b ;
    A = infsup(zeros(NN,1)) ;
    A(1) = a ;
    for k = 2:NN
        A(k) = a + (1/2) * exp(2 * A(k-1)) ;
    endfor
    alpha = inf(exp(A(NN))) ;
    B = infsup(zeros(NN,1)) ;
    B(1) = a + 1/2 ;
    for k = 2:NN
        B(k) = a + (1/2) * exp(2 * B(k-1)) ;
    endfor
    beta = sup(exp(B(NN))) ;
    if (beta <= alpha)
        printf("Attention: alpha >= beta at row:\n")
        n
    endif
    if (3*beta - 2*alpha > 1)
        printf("Attention: 3 alpha - 2 beta > 1 at row:\n")
        n
    endif
    % Verify: it should be:
    % inf(ver_a) > 0 and sup(ver_b) < 0.
    ver_a = 1/2 * infsup(alpha)^2 - log(infsup(alpha)) - 1/2 * R^2 + log(R) - log(sqrt(infsup(3)) / 2);
    ver_b = 1/2 * infsup(beta)^2 - log(infsup(beta)) - 1/2 * R^2 + log(R) - log(sqrt(infsup(3)) / 2);
    if (inf(ver_a) <= 0)
        printf("Attention: not accurate enclosure of d(R) at row:\n")
        n
    endif
    if (sup(ver_b) >= 0)
        printf("Attention: not accurate enclosure of d(R) at row:\n")
        n
    endif
    dR = infsup( alpha , beta );
    vec_dR(n) = dR;
    m_0(n) = floor( - log2(beta - alpha) );
    delta_0 = 2^(-m_0(n));
    if (delta_0 < beta - alpha)
        printf("Attention: something wrong, delta_0 too small at row:\n")
        n
    endif
    if (alpha == 0)
        printf("Attention: alpha = 0 at row:\n")
        n
    endif
    if (beta + delta_0 > 1)
        printf("Attention: beta + delta_0 > 1 at row:\n")
        n
        printf("You should take a larger NN and/or a smaller delta_R")
    endif
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % J_0
    p = dR + delta_0;
    fp = exp( p^2 / 2 - R^2 / 2) * R * (1/p)^2 * rsqrt( 1 - exp( p^2 - R^2 ) * R^2 * (1/p)^2 );
    J_0(n) = delta_0 / 2 * ( sqrt(infsup(3)) / dR + fp );
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % J_1
    N_1 = floor( (1 - alpha - delta_0) / delta_1 );
    x = zeros(N_1 + 1,1);
    x(1) = alpha + delta_0;
    for i = 2 : N_1+1

```

```

x(i) = x(i-1) + delta_1;
endifor
if (x(N_1 + 1) > 1)
  printf("Attention: x(N_1 + 1) > 1 at row:\n")
  n
endif
y = infsup(zeros(N_1+1,1));
y = exp( x.^2 / 2 - R^2 / 2) * R .* ((1./x).^2) .* rsqrt( 1 - exp( x.^2 - R^2 ) * R^2 .* ((1./x).^2) );
J_1(n) = delta_1 * (y(1) + y(N_1+1)) / 2 + delta_1 * sum( y(2 : N_1) );
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% J_4
J_4(n) = 2 * sqrt(infsup(delta_4)) * rsqrt(2*R - delta_4) * rsqrt((R - delta_4)^2 - 1);
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% J_3
N_3 = floor( (sup(R) - delta_4 - 1) / delta_3 );
z = zeros(N_3+1,1);
z(1) = sup(R) - delta_4;
for i = 2 : N_3+1
  z(i) = z(i-1) - delta_3;
endifor
if ( z(N_3 + 1) < 1 )
  printf("Attention: z(N_3 + 1) < 1 at row:\n")
  n
endif
w = infsup(zeros(N_3+1,1));
w = exp( z.^2 / 2 - R^2 / 2) * R .* ((1./z).^2) .* rsqrt( 1 - exp( z.^2 - R^2 ) * R^2 .* ((1./z).^2) );
J_3(n) = delta_3 * (w(1) + w(N_3 + 1)) / 2 + delta_3 * sum( w(2 : N_3) );
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%% J_2
if ( z(N_3 + 1) - x(N_1 + 1) >= delta_1 + delta_3 )
  printf("Attention: z(N_3 + 1) - x(N_1 + 1) should not be so large, at row:\n")
  n
endif
J_2(n) = ( y(N_1 + 1) + w(N_3 + 1) ) * ( z(N_3 + 1) - x(N_1 + 1) ) / 2;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
endifor
J = J_0 + J_1 + J_2 + J_3 + J_4;
[J_max, n_max] = max(sup(J));
elapsed_time = toc ();

M
printf("J_0,J_1,J_2,J_3,J_4 at max:\n")
[J_0(n_max); J_1(n_max); J_2(n_max); J_3(n_max); J_4(n_max)]

printf("R, d(R) at max:\n")
[vec_R(n_max); vec_dR(n_max)]

printf("Time in seconds:\n")
tt = elapsed_time;
tt % tt/60

printf("Min and max of d(R):\n")
min(inf(vec_dR))
max(sup(vec_dR))

printf("Min and max of m_0:\n")
min(m_0)
max(m_0)

printf("J max:\n")
J(n_max)

```

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(Pietro Baldi) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI NAPOLI FEDERICO II, VIA CINTIA, MONTE S. ANGELO, NAPOLI, ITALY, I-80126

E-mail address, P. Baldi: pietro.baldi@unina.it

(Emanuele Haus) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI NAPOLI FEDERICO II, VIA CINTIA, MONTE S. ANGELO, NAPOLI, ITALY, I-80126

E-mail address, E. Haus: emanuele.haus@unina.it

(Carlo Mantegazza) DIPARTIMENTO DI MATEMATICA E APPLICAZIONI, UNIVERSITÀ DI NAPOLI FEDERICO II, VIA CINTIA, MONTE S. ANGELO, NAPOLI, ITALY, I-80126

E-mail address, C. Mantegazza: c.mantegazza@sns.it