Ann. Henri Poincaré 19 (2018), 443–462

1424-0637/18/020443-20
published online November 30, 2017
https://doi.org/10.1007/s00023-017-0627-5

Annales Henri Poincaré

Effects of Boundary Conditions on Irreversible Dynamics

Aldo Procacci, Benedetto Scoppola and Elisabetta Scoppola

Abstract. We present a simple one-dimensional Ising-type spin system on which we define a completely asymmetric Markovian single spin-flip dynamics. We study the system at a very low, yet nonzero, temperature, and we show that for free boundary conditions the Gibbs measure is stationary for such dynamics, while introducing in a single site a + condition the stationary measure changes drastically, with macroscopical effects. We achieve this result defining an absolutely convergent series expansion of the stationary measure around the zero temperature system. Interesting combinatorial identities are involved in the proofs.

1. Introduction

In this paper, we discuss a very simple one-dimensional spin system in order to point out the crucial effect of boundary conditions on the invariant measure of irreversible dynamics.

Irreversible dynamics turn out to be a challenging problem since they are the main ingredient in the study of non-equilibrium statistical mechanics. Indeed many interesting physical systems cannot be described in terms of equilibrium: for instance non-Hamiltonian evolutions, systems with external non-conservative forces, or systems with thermostats or reservoirs. Such systems exhibit nonzero currents of matter or energy flowing in an *irreversible* way. For this kind of problem, it is necessary to consider non-equilibrium statistical mechanics. Actually we can say that the description of non-equilibrium systems represents one of the "grand challenges" in statistical mechanics.

In this frame, the main point is to describe *non-equilibrium stationary states* (NESS), "in understanding the properties of states which are in stationary non-equilibrium, thus establishing a clear separation between properties of evolution toward stationarity (or equilibrium) and properties of the stationary states themselves: a distinction which until the 1970's was rather blurred." as mentioned in the beautiful book by Gallavotti [7].

Irreversible dynamics play in this context a crucial role. The invariant measures of irreversible dynamics are stationary states, but they describe nonzero currents of probability, and hence they are NESS. A famous example is given by the TASEP model, in which particles hop only to the right, entering from a left reservoir with a given rate and leaving the system from the site L with another rate.

In the context of Markovian dynamics, given any two states i and j in some configuration space $\mathcal{X}$, the irreversibility is defined by transition probabilities $P(i,j)$ violating the detailed balance condition

$$\pi(j)P(j,i) = \pi(i)P(i,j) \qquad \forall i,j \in \mathcal{X}$$

This means that there are nonzero probability currents. Indeed given a pair of states $i,j \in \mathcal{X}$ define the probability current (or flow of probability) from j to i at time t the antisymmetric function on $\mathcal{X} \times \mathcal{X}$:

$$K_t(j,i) = P^t(j)P(j,i) - P^t(i)P(i,j)$$

where $P^t(\cdot)$ represents the probability of the state $\cdot$ at time t.

The continuity equation for $P^t(i)$ gives

$$\begin{aligned} P^{t+1}(i) - P^t(i) &= \sum_j P^t(j)P(j,i) - P^t(i)\sum_j P(i,j) \\ &= \sum_{j\neq i}\Big(P^t(j)P(j,i) - P^t(i)P(i,j)\Big) = \sum_{j\neq i} K_t(j,i) \\ &= -(\operatorname{div} K_t)(i) \end{aligned}$$

Stationarity implies

$$0 = \sum_{j\neq i}\Big(\pi(j)P(j,i) - \pi(i)P(i,j)\Big) = \sum_{j\neq i} K(j,i) \qquad \forall i \tag{1}$$

where $K(j,i) = \pi(j)P(j,i) - \pi(i)P(i,j)$ is the stationary probability current (or stationary flow of probability) from j to i, a divergence-free flow. This flow K is proportional to the antisymmetric part of the conductance associated with the chain, and it is also considered for instance in [8]. Actually the presence of currents can be used to detect irreversible dynamics without using the invariant measure. This is done by the Kolmogorov criterion for reversibility [9]: The Markov dynamics with transition probabilities $P(i,j)$ is reversible if and only if for any loop of states: $i_0, i_1, i_2, ..., i_n, i_0$ we have

$$P(i_0,i_1)P(i_1,i_2)\cdots P(i_n,i_0) = P(i_0,i_n)\cdots P(i_2,i_1)P(i_1,i_0).$$

This means that the dynamics is irreversible if there is a loop with a stationary current. Actually, it is natural to set the stationary probability of the sequence of states $i_0, i_1, \ldots, i_n, i_0$ as

$$\pi(i_0)P(i_0,i_1)P(i_1,i_2)\cdots P(i_n,i_0)$$

and so a stationary probability current

$$\pi(i_0)\Big[P(i_0,i_1)P(i_1,i_2)\cdots P(i_n,i_0) - P(i_0,i_n)\cdots P(i_2,i_1)P(i_1,i_0)\Big].$$

This means that at the loop $i_0, i_1, \ldots, i_n, i_0$, without a fixed starting point, we can associate a stationary probability current

$$\sum_{k=0}^{n} \pi(i_k)\Big[P(i_0,i_1)P(i_1,i_2)\cdots P(i_n,i_0) - P(i_0,i_n)\cdots P(i_2,i_1)P(i_1,i_0)\Big].$$

As noted in the rich review by Chou et al. [2], the presence of stationary current loops suggests to associate magnetostatics to irreversible dynamics as electrostatics is associated to reversible dynamics. This is an evocative and effective association since magnetostatics is related to stationary currents and electrostatics to stationary electric charges.

Beside their crucial role in the understanding of non-equilibrium statistical mechanics, irreversible dynamics have been frequently considered in the literature in order to speed up simulations. Indeed, in some cases, rigorous control of mixing time of irreversible dynamics has been obtained. See for instance [5].

Several problems arise when considering irreversible dynamics. Indeed some tools frequently used in the study of convergence to equilibrium are strongly related to reversibility, such as spectral representation or the potential theoretical approach. Recently, some progress has been made to extend some of these tools to non-reversible dynamics. See for instance the extension of the Dirichlet principle to non-reversible Markov chains obtained in [8].

In this paper we want to stress the main difficulty related to irreversibility: While detailed balance is a crucial tool to control the invariant measure of reversible dynamics, in the irreversible case the control of the invariant measure can be quite complicated, and in particular it is difficult to study its sensitivity to boundary conditions. Very recent results have been obtained in this direction in [6] where irreversible dynamics are constructed with a given Gibbsian stationary measure by exploiting cyclic decomposition of divergence-free flows.

In some cases, it is possible to verify that the equation for the invariant measure (1) is satisfied by a suitable Gibbs measure, as proved below in the (easy) case of free boundary conditions. This is also the case of the two-dimensional Ising model with asymmetric interaction and periodic boundary conditions discussed in [11]. In general, due to the presence of probability currents, the verification of Eq. (1) typically involves non-local argument; see Remark 3 below, and so the invariant measure strongly depends on boundary conditions.

We consider a one-dimensional spin system on the discrete interval $[1, L] \equiv \{1, 2, ..., L\}$ with a single spin-flip Markovian dynamics $\{X_t\}_{t\in\mathbb{N}}$, defined on $\mathcal{X} := \{-1, 1\}^L$ by the following transition probabilities

$$P(\sigma, \sigma^{(i)}) = \frac{1}{L}\mathrm{e}^{-2J(\sigma_i\sigma_{i-1}+1)} \tag{2}$$

where $\sigma^{(i)}$ is the configuration obtained from σ flipping the spin in the site $i \in \{1, 2, ..., L\}$. This means that at each time a site i is chosen uniformly at random in $\{1, 2, ..., L\}$ and the spin is flipped in this site with probability one if it is opposite to its left neighbor, σ_{i-1}, or with probability e^{-4J} if it is parallel to σ_{i-1}. We will consider two different boundary conditions:

- the free boundary condition corresponding to $\sigma_0 = 0$;
- the + boundary condition corresponding to $\sigma_0 = +1$.

The chain is irreducible and aperiodic so that in both cases there exists a unique invariant measure. We will consider a particular low-temperature regime, the *chilled regime*, (see definition below) where the inverse temperature J is proportional to $\log L$. This is of course a strong assumption that we will use in order to easily compute the invariant measure of the process. We stress again that our goal is just to point out that in irreversible dynamics the boundary effect does not act simply as a conditioning, like in the Gibbs measure, but changes completely the structure of the invariant measure.

To this end we shall prove that while in the case of free boundary conditions the stationary distribution is the Gibbs measure, in the case of + boundary condition the stationary measure changes drastically. Due to the particular low-temperature regime we are able to write the stationary distribution as an absolutely convergent expansion in e^{-4J}. This expansion is easily controlled in this case, but it could be a general tool in order to handle the invariant measure at a very low temperature in more general contexts. We control completely the first order of such expansion, and we show that it has several interesting features. The boundary conditions actually modify the stationary distribution and the effect of their presence decays very slowly in the distance i from the boundary, namely as $\frac{1}{\sqrt{i}}$. Moreover, the presence of boundary conditions makes the probabilities of interval of minus spins dependent on their length, producing macroscopical effects on the magnetization.

The paper is organized as follows: In Sect. 2 we define the models, comparing them with the usual reversible Glauber Dynamics for the 1d Ising Model, and we state the main results of the paper. Section 3 is devoted to the control of the expansion of the invariant measure in terms of the quantity e^{-4J}. Section 4 contains the proof related to the characterization of the first order term of the invariant measure. Some concluding remarks and future perspectives are discussed in Sect. 5.

2. Models and Results

As mentioned in the introduction, our model is defined via an irreversible markovian dynamics on a one-dimensional discrete spin chain with states $\sigma \in \mathcal{X} = \{-1,+1\}^{\{1,2,\ldots,L\}}$. We consider two different boundary conditions, namely the free boundary conditions, having $\sigma_0 = \sigma_{L+1} = 0$, and the + boundary condition $\sigma_0 = \sigma_{L+1} = +1$. The dynamics is defined by the following transition matrix

$$P^I(\sigma,\tau) = \begin{cases} \frac{1}{L}\mathrm{e}^{-2J(\sigma_i\sigma_{i-1}+1)} & \text{if } \tau = \sigma^{(i)} \\ 1 - \frac{1}{L}\sum_i \mathrm{e}^{-2J(\sigma_i\sigma_{i-1}+1)} & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $\sigma^{(i)}$ is the configuration obtained from σ by flipping the spin in the site i. We fix boundary conditions both in 0 and in $L+1$ in order to consider also the reversible case (see below) for comparison. However, we note that for the dynamics defined in (3) just the boundary condition in 0 acts on the dynamics. This dynamics is irreversible, but in the case of free boundary conditions, it is easy to find its stationary measure. Indeed, consider the Gibbs measure

$$\pi^G(\sigma) = \frac{\mathrm{e}^{-H(\sigma)}}{Z^G}, \qquad Z^G = \sum_{\sigma \in \mathcal{X}} \mathrm{e}^{-H(\sigma)} \tag{4}$$

where $H(\sigma)$ is the usual Ising Hamiltonian with free boundary conditions.

$$H(\sigma) = -J \sum_{i=2}^{L} \sigma_i \sigma_{i-1} \tag{5}$$

Let us show that $\pi^G(\sigma)$ is the unique stationary measure of dynamics (3). The dynamics is clearly irreducible and aperiodic, and hence the stationary measure exists and it is unique.

Moreover, it is immediate to verify the following equalities:

$$\pi^G(\sigma^{(i)}) = \pi^G(\sigma)\mathrm{e}^{-2J(\sigma_i\sigma_{i-1}+\sigma_i\sigma_{i+1})} \tag{6}$$

$$P^I(\sigma^{(i)}, \sigma) = \frac{1}{L}\mathrm{e}^{2J(\sigma_i\sigma_{i-1}-1)}. \tag{7}$$

To prove that π^G is the invariant measure of the process satisfying

$$\sum_{\tau \in \mathcal{X}} \pi^G(\tau) P^I(\tau, \sigma) = \pi^G(\sigma), \tag{8}$$

it is sufficient to verify the following condition, obtained from (8) by canceling the diagonal terms in both sides of the equality, which is equivalent to Eq. (1):

$$\sum_{i=1}^{L} \pi^G(\sigma^{(i)}) P^I(\sigma^{(i)}, \sigma) = \pi^G(\sigma) \sum_{i=1}^{L} P^I(\sigma, \sigma^{(i)}) \tag{9}$$

Equation (9) immediately follows from (6) and (7) since, by the free b.c, we have

$$\sum_{i=1}^{L} \mathrm{e}^{-2J(\sigma_i\sigma_{i+1})} = \sum_{i=1}^{L-1} \mathrm{e}^{-2J(\sigma_i\sigma_{i+1})} = \sum_{i=2}^{L} \mathrm{e}^{-2J(\sigma_{i-1}\sigma_i)}.$$

It is a standard task to define a reversible markovian dynamics having the same stationary measure, i.e., the well-known *Glauber dynamics*, given by the following transition probability matrix

$$P^R(\sigma, \tau) = \begin{cases} \frac{1}{L}\mathrm{e}^{-[H(\sigma^{(i)})-H(\sigma)]_+} & \text{if } \tau = \sigma^{(i)} \\ 1 - \sum_i \frac{1}{L}\mathrm{e}^{-[H(\sigma^{(i)})-H(\sigma)]_+} & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

where $[\cdot]_+$ means the positive part.

For both dynamics, the one-dimensional stationary measure $\pi^G(\sigma)$ is well known. We have

$$\pi^G(\sigma) = \frac{\mathrm{e}^{-2J\ell(\sigma)}}{2\Big(1+\mathrm{e}^{-2J}\Big)^{L-1}}$$

where $\ell(\sigma)$ is the number of pair $\{i, i+1\}$ such that $\sigma_i\sigma_{i+1} = -1$ (i.e., $\ell(\sigma)$ is the total length of the Peierls contours).

We conclude this short discussion of the free boundary conditions by checking the irreversibility of the dynamics defined in (3), i.e., the presence of nonzero probability currents. Indeed, for example, for $i > 1$ and $m > 1$ such that $i + m < L$, let us consider the configuration σ with $\sigma_j = -1$ for $j = i, i+1, \ldots, i+m-1$ and $\sigma_j = +1$ elsewhere and observe that $\pi^G(\sigma) = \pi^G(\sigma^{(i)})$ while $P(\sigma, \sigma^{(i)}) = \frac{1}{L}$ and $P(\sigma^{(i)}, \sigma) = \frac{\mathrm{e}^{-4J}}{L}$. Therefore,

$$\pi^G(\sigma)P(\sigma, \sigma^{(i)}) - \pi^G(\sigma^{(i)})P(\sigma^{(i)}, \sigma) = \Big[1 - \mathrm{e}^{-4J}\Big]\frac{\pi^G(\sigma)}{L} > 0.$$

The effect of irreversibility clearly appears looking at typical trajectories of the dynamics, where the interface between spins with different orientations, i.e., the Peierls contours, are moving rightward.

In order to control the invariant measure in the case of plus boundary conditions, we introduce a particular regime, which is assumed throughout the paper, defined as follows.

Definition of chilled regime
We say that the one-dimensional discrete spin chain on $[1, L]$ with states $\sigma \in \{-1,+1\}^{\{1,\ldots,L\}}$ subjected to the irreversible dynamics (3) or to the Glauber dynamics (10) is in the *chilled regime of parameter* $c > 0$ *if*

$$J = c \log L$$

Note that the Gibbs measure π^G for c large enough is concentrated on the configurations $\sigma = \boxplus$ ($\sigma_i = 1\ \forall i$) and $\sigma = \boxminus$ ($\sigma_i = -1\ \forall i$), while for the other configurations σ, we get

$$\pi^G(\sigma) \sim \frac{1}{2}\mathrm{e}^{-2J\ell(\sigma)}$$

The chilled condition defined above mimics a phase transition, in the sense that the volume-dependent low temperature (high J) defined by $\mathrm{e}^{-2J}L \ll 1$ forces the system in a nonzero (in particular, very close to ±1) magnetization. It is very easy, yet quite interesting, to study the mixing time of the two dynamics defined above, which is proportional to the expected value of the *tunneling time*, namely the time needed to pass from the configuration $\boxplus$ to the configuration $\boxminus$.

It is not difficult to identify in the reversible case the typical path of the tunneling. By chilled condition $\mathrm{e}^{-2J}L \ll 1$, a spin flip on the boundary occurs after a time of the order of $L\mathrm{e}^{2J}$ and a spin flip inside a region of spins having all the same sign occurs after a time of the order of e^{4J}. Both times are much longer than L, and the shorter one, giving the typical path of the tunneling, is the first, $L\mathrm{e}^{2J}$. The interface between two regions with opposite spins may move in a time of order L, with equal probability on the right and on the left. A random walk of such surface reaches eventually the other boundary with probability 1/L in a number of steps of order L^2, and hence in a time of order L^3. Hence the expected time that one has to wait for the tunneling, since typically one has to try L times before the random walk hits the opposite boundary, is $L^2\mathrm{e}^{2J}$ plus the time L^3, which has to be added to the previous one. The leading order of such time is hence $L^2\mathrm{e}^{2J}$. which is precisely the tunneling time. In the irreversible dynamics, the spin in the site 1 is flipped after a time of the order $L\mathrm{e}^{2J}$. The boundary between the + and the − regions, then, typically moves only on the right, doing L steps, each of which is performed in a time of the order L, giving a total time of the order L^2. Again, this ensures that the tunneling time is of the order of $L\mathrm{e}^{2J}$, because the (added) time L^2 is smaller than $L\mathrm{e}^{2J}$. Hence the tunneling time in the irreversible case is shorter, polynomially in L, than the one in the reversible case.

In what follows, we will consider the case of + boundary conditions, namely $\sigma_0 = \sigma_{L+1} = 1$. With the reversible Glauber dynamics (10), the invariant measure with plus boundary conditions is just Gibbs measure π^G conditioned to $\sigma_0 = \sigma_{L+1} = 1$. If we consider now the irreversible dynamics (3), we will see ahead that its invariant measure changes dramatically with respect to the free boundary conditions case.

For notational simplicity, we will also use the notation $P_{\sigma\tau} \equiv P(\sigma,\tau)$ and $\pi_\sigma \equiv \pi(\sigma)$ in computations below.

2.1. Results

Before stating our results concerning this particular regime, we need to introduce the main technical tool which consists in writing the invariant measure of the irreversible dynamics with + boundary conditions in the chilled regime $J = c\log L$ in terms of a series in e^{-4J}. We will omit for simplicity hereafter the suffix I, standing for irreversibility.

Denoting with $\ell(\sigma)$ the number of antiparallel adjacent pairs of spins for each configuration σ and recalling that $\sigma_0 = 1$, we can write the transition probability matrix in the following form

$$P(\sigma,\tau) = \begin{cases} \frac{1}{L} & \text{if } \tau = \sigma^{(i)} \text{ and } \sigma_i\sigma_{i-1} = -1 \\ \frac{\mathrm{e}^{-4J}}{L} & \text{if } \tau = \sigma^{(i)} \text{ and } \sigma_i\sigma_{i-1} = 1 \\ 1 - \frac{\ell(\sigma)}{L} - \left(1 - \frac{\ell(\sigma)}{L}\right)\mathrm{e}^{-4J} & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

We can define the dynamics above for zero temperature ($J \to \infty$)

$$P^{(0)}(\sigma,\tau) = \begin{cases} \frac{1}{L} & \text{if } \tau = \sigma^{(i)} \text{ and } \sigma_i\sigma_{i-1} = -1 \\ 1 - \frac{\ell(\sigma)}{L} & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases}$$

obtaining

$$P(\sigma,\tau) = P^{(0)}(\sigma,\tau) + \mathrm{e}^{-4J}\Delta P(\sigma,\tau) \tag{12}$$

where

$$\Delta P(\sigma,\tau) = \begin{cases} \frac{1}{L} & \text{if } \tau = \sigma^{(i)} \text{ and } \sigma_i\sigma_{i-1} = 1 \\ -1 + \frac{\ell(\sigma)}{L} & \text{if } \tau = \sigma \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

The state σ corresponding to $\ell(\sigma) = 0$, i.e., $\sigma = \boxplus$ ($\sigma_i = +1\ \forall\ i$), is clearly absorbent for the zero-temperature dynamics. Hence

$$\pi^{(0)}(\sigma) = \begin{cases} 1 & \text{if } \sigma = \boxplus \\ 0 & \text{otherwise} \end{cases}$$

We can use now the following formula for the perturbations on Markov chains:

$$\pi(\sigma) = \sum_{k=0}^{\infty} \mathrm{e}^{-4Jk}\pi^{(k)}(\sigma) \tag{14}$$

where

$$\pi^{(k)}(\sigma) = \sum_{\tau} \pi^{(0)}(\tau) D^k(\tau,\sigma) \qquad D = \sum_{j=0}^{\infty} \Delta P (P^{(0)})^j \tag{15}$$

Again for notational simplicity, we will write $P^{(0)j} \equiv (P^{(0)})^j$. Note that by its definition

$$\pi^{(k)}(\sigma) = 0 \quad \forall\ \sigma:\ \ell(\sigma) > 2k \tag{16}$$

Formulas (14) and (15) may be easily proved in general. Indeed, let $\pi_i^{(0)}$ the stationary measure of a Markov chain $P_{ij}^{(0)}$. Consider the ergodic chain $P_{ij} = P_{ij}^{(0)} + \varepsilon\Delta P_{ij}$. Denote with π_i the stationary measure of the chain P_{ij}. By ergodic theorem, we have

$$\pi_i = \lim_{N\to\infty} \sum_j \pi_j^{(0)} (P_{ij})^N = \lim_{N\to\infty} \sum_j \pi_j^{(0)} (P_{ij}^{(0)} + \varepsilon\Delta P_{ij})^N$$

Then defining

$$D_{ij} = \sum_{l\geq 0} \sum_k \Delta P_{ik} (P^{(0)l})_{kj},$$

we have that

$$\pi_i = \sum_k \pi_i^{(k)} \varepsilon^k$$

with

$$\pi_i^{(k)} = \sum_l \pi_l^{(0)} (D^k)_{li}$$

A similar expansion is used for instance in [3] for the blockage problem.

We define the expansion of the stationary measure up to the first order as

$$\pi^{(\leq 1)} = \pi^{(0)} + \mathrm{e}^{-4J}\pi^{(1)} \tag{17}$$

Note that $\pi^{(\leq 1)}$ is a probability measure. We want to remark that at zero order there are no currents because the system is frozen in the configuration $\boxplus$. But at each subsequent order, it is immediate to see that currents are present and the dynamics is irreversible (see Remark 3 below).

We can now state our main results. The first is an immediate consequence of the convergence of the perturbative expansion (14). Let

$$d_{TV}(\pi, \pi^{(\leq 1)}) = \sum_{\sigma} \left| \pi(\sigma) - \pi^{(\leq 1)}(\sigma) \right|$$

be the total variation distance between the measure π and its first-order approximation $\pi^{(\leq 1)}$. Then the following theorem holds.

Theorem 2.1. *In chilled regime of parameter $c = \frac{1}{2} + \gamma$, with $\gamma > 0$, we have that*

$$d_{TV}(\pi, \pi^{(\leq 1)}) \leq \frac{const}{L^{8\gamma}} \tag{18}$$

Theorem 2.1 shows that it is meaningful, in the chilled regime with $\gamma > 1/2$, to compute the first order in e^{-4J} of the stationary measure, since it will be the leading one.

As it is clear from the perturbative approach, by (16), up to first order the only configurations admitted are the ones with at most one connected interval of sites having $\sigma_i = -1$, while all the rest of the configuration has $\sigma_i = +1$.

Remark 1 The first-order perturbative expansion introduced above underlines another way to see the impact of boundary conditions on irreversible dynamics. In the first-order picture the only possibility to construct an interval of $-$ spins in the sea of $+$ spins, of a given length and in a given position, is to flip a single $-$ spin and then to let evolve the system according to the zero-temperature dynamics until such evolution covers exactly the desired interval. The sum of this kind of evolutions is exactly the meaning of the matrix D_{ij} defined above, and hence this sum is a way to define the stationary measure up to the first order. It happens that with free boundary conditions (or, as it is easy to see, on a ring, with periodic boundary conditions), this picture of the stationary measure in terms of sum over trajectories is equivalent to the (more familiar) Gibbs measure. Fixing a $+$ boundary condition the number of possible trajectories is drastically cut when the $-$ interval is close to the boundary, and then this equivalence is lost. This is due to the fact that Peierls contours at zero temperature move only rightward. The Gibbs measure, in this case, has nothing to share with the stationary distribution of the system.

Let $i \in 1, \ldots, L-1$ and $m \in 1, \ldots, L-i$ and let us denote $(i;m)$ the state having

$$\sigma_k = \begin{cases} +1 & \text{for } 1 \le k < i \\ -1 & \text{for } i \le k < i+m \\ +1 & \text{for } i+m \le k \le L \end{cases}$$

In other words $\sigma_i\sigma_{i-1} = -1$, $\sigma_{i+m}\sigma_{i+m-1} = -1$, $\sigma_k\sigma_{k-1} = 1$ $\forall k \neq i, i+m$. That is, the state $(i;m)$ is a single interval of m spins equal to -1 starting at i.

Let us denote (i) the state having

$$\sigma_k = \begin{cases} +1 & \text{for } 1 \le k < i \\ -1 & \text{for } i \le k \le L \end{cases}$$

In other words $(i) = (i; L+1-i)$, i.e., $\sigma_i\sigma_{i-1} = -1$, $\sigma_k\sigma_{k-1} = 1$ $\forall k \neq i$.

Theorem 2.2. *For any fixed $m > 0$ and i large, we have*

$$\pi^{(\le 1)}_{(i;m)} = \mathrm{e}^{-4J}\left(1 - \frac{C_m}{\sqrt{i}} + o\Big(\frac{1}{\sqrt{i}}\Big)\right) \tag{19}$$

where C_m is a constant depending on m. For every i, m we have

$$\pi^{(\le 1)}_{(i;m)} \le 4\,\mathrm{e}^{-4J}\ \mathrm{e}^{-\frac{(m)^2}{2(i+m)}}\ m \tag{20}$$

Moreover, for every i

$$\pi^{(\le 1)}_{(i)} = \sum_{l=1}^{i} \pi^{(\le 1)}_{(l;L-l)} \tag{21}$$

Remark 2 Note that (19) gives a precise quantitative meaning to the comments in Remark 1: we get $\pi^{(\le 1)}_{(i;m)} \to \mathrm{e}^{-4J}$ as $i \to \infty$, so that very far from the boundary condition the stationary distribution at the first order in e^{-4J} is equal to the Gibbs one, giving the same weight to every interval of minus spins independently of its length and its position. This convergence to the Gibbs measure, however, is very slow, and it does not occur on a well-defined scale. Moreover, the exponential decay with the length m of the interval of $-$ spins given by (20) produces macroscopic effects, as the following theorem shows.

Theorem 2.3. *The average value of $m(\sigma) := \sum_{i=1}^{L} \mathbb{1}_{\{\sigma_i=-1\}}$ with respect to the Gibbs measure, π^G, and with respect to the irreversible measure up to the first order, $\pi^{(\le 1)}$, is given by*

$$\lim_{L\to\infty} \frac{\pi^{(\le 1)}(m)}{\pi^G(m)} \le \frac{1}{4} \tag{22}$$

Remark 3 Looking at simulations the presence of a current in the system up to the first order in e^{-4J}, as discussed above, is absolutely evident: The typical evolution is given by a single interval of $-$ spins moving rightward. The irreversibility is, up to this order, crystal clear.

3. Proof of Theorem 2.1

By (15) we have

$$d_{TV}(\pi,\pi^{(\le 1)}) = \sum_\sigma \left|\sum_{k=2}^{\infty} e^{-4Jk}\pi_\sigma^{(k)}\right| \le \sum_{k=2}^{\infty} e^{-4Jk}\sum_\sigma \left|\pi_\sigma^{(k)}\right|$$

For $J = c\log L$ the condition $c = \frac{1}{2}+\gamma$ implies $e^{-4Jk} = L^{-(2+4\gamma)k}$ and then it is enough to prove that

$$\sum_\sigma \left|\pi_\sigma^{(k)}\right| \le (CL^2)^k \tag{23}$$

Since

$$\sum_\sigma \left|\pi_\sigma^{(k)}\right| = \sum_\sigma \left|\sum_{m=0}^{\infty}\sum_{\tau,\sigma'} \pi_\tau^{(k-1)}\Delta P_{\tau\sigma'}\left(P^{(0)m}\right)_{\sigma'\sigma}\right|$$

we have that (23) is recursively proved if we are able to prove that

$$\sup_\tau \sum_\sigma \left|\sum_{m=0}^{\infty}\sum_{\sigma'} \Delta P_{\tau\sigma'}\left(P^{(0)m}\right)_{\sigma'\sigma}\right| \le CL^2 \tag{24}$$

Note first that

$$\sum_{\sigma'} \Delta P_{\tau\sigma'} = 0 \tag{25}$$

for all τ. Then define the matrix $\Pi^{(0)}$, having all the rows equal to the stationary measure $\pi^{(0)}$, and hence having on the column related to the configuration $\sigma = \boxplus$, say on the first column, all the entries equal to 1, while all the other entries are zero. Observe that, due to (25), we have

$$\sum_{\sigma'} \Delta P_{\tau\sigma'}\Pi^{(0)}_{\sigma',\tau} = 0 \tag{26}$$

for all σ and τ. Finally define

$$R_m = P^{(0)m} - \Pi^{(0)} \tag{27}$$

Due to (26) we have that

$$\sum_{\sigma'} \Delta P_{\tau\sigma'}\left(P^{(0)m}\right)_{\sigma'\sigma} = \sum_{\sigma'} \Delta P_{\tau\sigma'}\left(R_m\right)_{\sigma'\sigma} \tag{28}$$

Now using (28) we split the sum over m in two terms:

$$\begin{aligned}\sum_\sigma &\left|\sum_{m=0}^{\infty}\sum_{\sigma'} \Delta P_{\tau\sigma'}\left(P^{(0)m}\right)_{\sigma'\sigma}\right| \\ &\le \sum_\sigma \left|\sum_{\sigma'} \Delta P_{\tau\sigma'}\left(\sum_{m=0}^{2L^2} P^{(0)m}\right)_{\sigma'\sigma}\right| + \sum_\sigma \left|\sum_{\sigma'} \Delta P_{\tau\sigma'}\left(\sum_{m=2L^2+1}^{\infty} R_m\right)_{\sigma'\sigma}\right|\end{aligned} \tag{29}$$

The first term is estimated as follows

$$\sum_{\sigma}\left|\sum_{\sigma'}\Delta P_{\tau\sigma'}\left(\sum_{m=0}^{2L^2}P^{(0)m}\right)_{\sigma'\sigma}\right| \le \sum_{\sigma,\sigma'}\left|\Delta P_{\tau\sigma'}\left(\sum_{m=0}^{2L^2}P^{(0)m}\right)_{\sigma'\sigma}\right|$$
$$\le \sum_{\sigma'}|\Delta P_{\tau\sigma'}|\sum_{m=0}^{2L^2}\sum_{\sigma}\left(P^{(0)m}\right)_{\sigma'\sigma}$$

For each m, the sum over σ is 1, and then

$$\sum_{\sigma}\left|\sum_{\sigma'}\Delta P_{\tau\sigma'}\left(\sum_{m=0}^{2L^2}P^{(0)m}\right)_{\sigma'\sigma}\right| \le \sum_{\sigma'}|\Delta P_{\tau\sigma'}|\,(2L^2+1)$$

Due to the definition of $\Delta P_{\tau\sigma'}$, we have

$$\sum_{\sigma'}|\Delta P_{\tau\sigma'}| = 2(1-\frac{\ell(\tau)}{L}) \le 2; \tag{30}$$

we obtain the following estimate

$$\sum_{\sigma}\left|\sum_{\sigma'}\Delta P_{\tau\sigma'}\left(\sum_{m=0}^{2L^2}P^{(0)m}\right)_{\sigma'\sigma}\right| \le 4L^2+2 \tag{31}$$

Now we are left with the estimate of the second term in (29):

$$\sum_{\sigma}\left|\sum_{\sigma'}\Delta P_{\tau\sigma'}\left(\sum_{m=2L^2+1}^{\infty}R_m\right)_{\sigma'\sigma}\right| \le \sum_{\sigma,\sigma'}\left|\Delta P_{\tau\sigma'}\left(\sum_{m=2L^2+1}^{\infty}R_m\right)_{\sigma'\sigma}\right|$$

Let us first of all consider the entries of the matrix R_m. Calling $T_{\boxplus}(\sigma')$ the hitting time to the state $\boxplus$ starting from the state σ', we have that, since $\boxplus$ is an absorbent state,

$$(R_m)_{\sigma',\boxplus} = P^{(0)m}_{\sigma',\boxplus} - 1 = -P(T_{\boxplus}(\sigma') > m)$$

For the same reason

$$\sum_{\sigma\neq\boxplus}(R_m)_{\sigma',\sigma} = P(T_{\boxplus}(\sigma') > m)$$

and therefore

$$\sum_{\sigma}|(R_m)_{\sigma',\sigma}| = 2P(T_{\boxplus}(\sigma') > m)$$

Hence

$$\sum_{\sigma,\sigma'}\left|\Delta P_{\tau\sigma'}\left(\sum_{m=2L^2+1}^{\infty} R_m\right)_{\sigma'\sigma}\right| \leq 2\sum_{\sigma'}\left|\Delta P_{\tau\sigma'}\sum_{m=2L^2+1}^{\infty} P(T_{\boxplus}(\sigma')>m)\right|$$
$$\leq 2\left(\sup_{\sigma'}\sum_{m=2L^2+1}^{\infty} P(T_{\boxplus}(\sigma')>m)\right)\sum_{\sigma'}|\Delta P_{\tau\sigma'}|$$
$$\leq 4\sup_{\sigma'}\sum_{m=2L^2+1}^{\infty} P(T_{\boxplus}(\sigma')>m)$$

where in the last line we used again (30).

We are left with an estimate of the quantity $P(T_{\boxplus}(\sigma')>m)$ uniformly in σ'. Recall that the (zero temperature) dynamics chooses u.a.r. a site and tries to update it. Call ξ_1 the time needed to choose for the first time the site 1, then ξ_2 the time needed, after the first choice of the site 1, to choose for the first time the site 2, and so on so forth. Calling $\xi=\sum_{i=1}^{L}\xi_i$ we have that $\xi \geq T_{\boxplus}(\sigma')$ for all σ' . This is granted by the fact that after the time ξ_1, we have definitively that $\sigma_1=+1$, and after the time $\xi_1+\xi_2$, we have definitively that $\sigma_1=\sigma_2=+1$ and so on. Hence we have for all σ'

$$P(T_{\boxplus}(\sigma')>m)\leq P(\xi>m)$$

Being ξ_i a geometrical variable of probability $p=\frac{1}{L}$, and hence having $E(\xi_i)=L$, $Var(\xi_i)=L^2$ for all i, we have that ξ is the sum of L independent geometric identical variables, and therefore $E(\xi)=L^2$, $Var(\xi)=L^3$.

By Chebyshev inequality

$$P(\xi>m)=P(\xi-E(\xi)>m-E(\xi))=P(\xi-E(\xi)>m-L^2)\leq\frac{L^3}{(m-L^2)^2}$$

We have then proved that

$$\sup_{\sigma'}\sum_{m=2L^2+1}^{\infty} P(T_{\boxplus}(\sigma')>m)\leq\sum_{m=2L^2+1}^{\infty}\frac{L^3}{(m-L^2)^2}\leq L$$

which finally gives

$$\sum_{\sigma}\left|\sum_{\sigma'}\Delta P_{\tau\sigma'}\left(\sum_{m=2L^2+1}^{\infty} R_m\right)_{\sigma'\sigma}\right|\leq 4L \tag{32}$$

Combining (32) and (31) we get (24).

4. Proof of Theorems 2.2 and 2.3

Let us denote with $\lambda((k;1),(i;m))$ a sequence of spin flips, allowed by the zero temperature dynamics, that brings the configuration $(k;1)$ into the configuration $(i;m)$. Since at least one $-$ spin has to be present in all the steps of the sequence, the latter can be described by partial Dyck words, and the number of such sequence is given by the elements of the so-called *Catalan's triangle* (see, e.g., [1,13]).

We have

$$\begin{aligned}
\pi^{(1)}_{(i;m)} &= D_{+,(i;m)} = \frac{1}{L}\sum_{k=1}^{i}\sum_{s=0}^{\infty} P^{(0)s}_{(k;1),(i;m)} = \frac{1}{L}\sum_{k=1}^{i}\sum_{s=2i+m-2k-1}^{\infty} P^{(0)s}_{(k;1),(i;m)} \\
&= \frac{1}{L}\sum_{k=1}^{i}\frac{1}{L^{2i+m-2k-1}}\sum_{\lambda((k;1),(i;m))}\sum_{s'=0}^{\infty}\binom{2i+m-2k-1+s'}{s'}\left(1-\frac{2}{L}\right)^{s'} \\
&= \sum_{k=1}^{i}\frac{1}{L^{2i+m-2k}}\left(\frac{L}{2}\right)^{2i+m-2k} C_{i+m-k-1,i-k} \\
&= \sum_{k=1}^{i}\left(\frac{1}{2}\right)^{2i+m-2k} C_{i+m-k-1,i-k}, \qquad (33)
\end{aligned}$$

where in the second line we defined $s' = s - 2i - m + 2k + 1$, and in the third line we used the Taylor expansion, convergent for $|\alpha| < 1$, of the function $\left(\frac{1}{1-\alpha}\right)^{N+1}$

$$\left(\frac{1}{1-\alpha}\right)^{N+1} = \sum_{s=0}^{\infty}\binom{N+s}{s}\alpha^s.$$

In Eq. (33), $C_{i+m-k-1,i-k}$ denotes the number appearing in the position $i+m-k-1, i-k$ of the Catalan's triangle, i.e.,

$$C_{n,k} = \frac{(n+k)!(n-k+1)}{k!(n+1)!}. \qquad (34)$$

Calling $l = i - k$, we have

$$\pi^{(1)}_{(i;m)} = \sum_{l=0}^{i-1}\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} \qquad (35)$$

We will now prove the following lemma.

Lemma 4.1. *For every positive integer m, we have*

$$\sum_{l=0}^{\infty}\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} = 1 \qquad (36)$$

Proof. The quantity $\pi^{(1)}_{(i;m)}$ can be written in terms of a one-dimensional symmetric random walk (SRW), $S_n = \sum_{i=1}^{n} X_i$, with X_i independent Bernoulli variables $X_i \in \{-1, +1\}$. Indeed $C_{l+m-1,l}$ is the number of paths of the random walk $\{S_n\}_{n\in\mathbb{N}}$ such that $S_1 = 1$, $S_{2l+m} = m$ and $S_n > 0$ for any $n = 1, ..., 2l+m$. For the duality principle for random walks, we have that $(X_1, X_2, ..., X_n)$ has the same distribution of $(X_n, X_{n-1}, ..., X_1)$, so that the path $(0, S_1, S_2, ..., S_n)$ has the same probability of the time reversal path $(0, S_n - S_{n-1}, S_n - S_{n-2}, ..., S_n - 0)$. This implies that by denoting with τ_m

the first hitting time to m for the random walk starting at 0, we have for every positive integer m

$$\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} = P(\tau_m = 2l+m) \tag{37}$$

so that

$$\pi^{(1)}_{(i;m)} = \sum_{l=0}^{i-1} P(\tau_m = 2l+m) = P(\tau_m < 2i+m). \tag{38}$$

Formula (36) now immediately follows from (38) since for the SRW the hitting of any state is finite with probability one. □

Remark. The proof of (36) can also be obtained in a purely combinatorial framework. See for instance Lemma 18 in reference [10].

We now prove (19). From (35) and Lemma 4.1, we have

$$\pi^{(1)}_{(i;m)} = 1 - \sum_{l=i}^{\infty} \left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l}, \tag{39}$$

with

$$\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} = \left(\frac{1}{2}\right)^{2l+m} \frac{(2l+m)!}{(l+m)!l!}\,\frac{m}{2l+m}.$$

Using upper and lower Stirling's bounds for the factorials [12] valid for all $n \geq 1$

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \mathrm{e}^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \mathrm{e}^{\frac{1}{12n}},$$

we have, for any $l \geq 1$ and any $m \geq 1$,

$$\begin{aligned}
\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} &\leq \frac{\mathrm{e}^{\frac{1}{12}}}{\sqrt{2\pi}} \frac{\left(1+\frac{m}{2l}\right)^{2l+m}}{\left(1+\frac{m}{l}\right)^{l+m}} \frac{m}{\sqrt{l(l+m)(2l+m)}} \\
&\leq \frac{\mathrm{e}^{\frac{1}{12}}}{\sqrt{2\pi}} \left(\frac{1+\frac{m}{2l}}{1+\frac{m}{l}}\right)^m \left(\frac{\left(1+\frac{m}{2l}\right)^2}{1+\frac{m}{l}}\right)^l \frac{m}{\sqrt{l(l+m)(2l+m)}} \\
&\leq \frac{\mathrm{e}^{\frac{1}{12}}}{\sqrt{2\pi}} \left(\frac{l+\frac{m}{2}}{l+m}\right)^m \left(1+\frac{m}{l}\right) \frac{m}{\sqrt{l(l+m)(2l+m)}} \\
&= \frac{\mathrm{e}^{\frac{1}{12}}}{\sqrt{2\pi}} \left(1-\frac{m}{2(l+m)}\right)^m \frac{m}{l^{3/2}} \sqrt{\frac{l+m}{2l+m}} \\
&\leq \frac{1}{2}\,\mathrm{e}^{\frac{-m^2}{2(m+l)}}\,\frac{m}{l^{3/2}},
\end{aligned}$$

where in the last line we have used the trivial bound $(1-x) \leq \mathrm{e}^{-x}$ valid for all $x \geq 0$. Hence, for any $l \geq 1$ and any $m \geq 1$, we may roughly bound

$$\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} \leq \frac{m}{2}\,\frac{1}{l^{3/2}}. \tag{40}$$

A similar computation gives, for any $l \ge 1$ and any $m \ge 1$,

$$\begin{aligned}\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} &\ge \frac{\mathrm{e}^{-\frac{1}{6}}}{\sqrt{2\pi}} \frac{\left(1+\frac{m}{2l}\right)^{2l+m}}{\left(1+\frac{m}{l}\right)^{l+m}} \frac{m}{\sqrt{l(l+m)(2l+m)}}\\
&\ge \frac{1}{3}\left(\frac{1+\frac{m}{2l}}{1+\frac{m}{l}}\right)^{m}\left(\frac{\left(1+\frac{m}{2l}\right)^{2}}{1+\frac{m}{l}}\right)^{l} \frac{m}{\sqrt{l(l+m)(2l+m)}}\\
&\ge \frac{1}{3}\left(\frac{l+\frac{m}{2}}{l+m}\right)^{m} \frac{m}{\sqrt{l(l+m)(2l+m)}}\end{aligned}$$

Therefore, we may roughly bound for any $l \ge 1$ and any $m \ge 1$

$$\left(\frac{1}{2}\right)^{2l+m} C_{l+m-1,l} \ge \frac{2^{-m}}{3\sqrt{6}} \frac{1}{l^{3/2}} \tag{41}$$

From inequalities (40) and (41), the first statement (19) of Theorem 2.2 immediately follows.

In order to show (20), we write

$$\pi^{(1)}_{(i;m)} = \left(\frac{1}{2}\right)^{m} + \sum_{l=1}^{i-1}\left(\frac{1}{2}\right)^{2l+m} \frac{(2l+m)!}{(l+m)!l!} \frac{m}{2l+m}$$

Using now (40) and recalling that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \zeta(3/2) \le 3$, we get

$$\begin{aligned}\pi^{(1)}_{(i;m)} &\le \left(\frac{1}{2}\right)^{m} + \frac{1}{2}\sum_{l=1}^{i-1} \mathrm{e}^{\frac{-m^2}{2(m+l)}} \frac{m}{l^{3/2}} \le \left(\frac{1}{2}\right)^{m} + \frac{\mathrm{e}^{\frac{-m^2}{2(m+i)}}}{2}\sum_{l=1}^{\infty}\frac{m}{l^{3/2}}\\
&\le \left(\frac{1}{2}\right)^{m} + 3m\mathrm{e}^{\frac{-m^2}{2(m+i)}} \le (1+3m)\mathrm{e}^{\frac{-m^2}{2(m+i)}} \le 4\,m\,\mathrm{e}^{\frac{-m^2}{2(m+i)}}\end{aligned} \tag{42}$$

and inserting (42) inequality into (39) we get (20).

The computation of $\pi^{(1)}_{(i)}$ is similar, but it is necessary to choose the time in which the spin in the site L is flipped to $\sigma_L = -1$. We have

$$\begin{aligned}\pi^{(1)}_{(i)} &= D_{+,(i)} = \frac{1}{L}\sum_{k=1}^{i}\sum_{m=0}^{\infty} P^{(0)m}_{(k;1),(i)}\\
&= \frac{1}{L}\sum_{k=1}^{i}\sum_{l=k}^{i}\frac{1}{L^{L+l-2k-1}}\sum_{\lambda((k;1),(l;L-l))}\sum_{m'=0}^{\infty}\binom{L+l-2k-1+m'}{m'}\\
&\quad\times\left(1-\frac{2}{L}\right)^{m'}\frac{1}{L}\frac{1}{L^{i-l}}\sum_{m''=0}^{\infty}\binom{i-l+m''}{m''}\left(1-\frac{i}{L}\right)^{m''}\\
&= \sum_{k=1}^{i}\sum_{l=k}^{i}\left(\frac{1}{2}\right)^{L+l-2k} C_{L-k-1,l-k} = \sum_{l=1}^{i}\sum_{k=1}^{l}\left(\frac{1}{2}\right)^{L+l-2k} C_{L-k-1,l-k}\\
&= \sum_{l=1}^{i}\pi^{(1)}_{(l;L-l)}\end{aligned}$$

This ends the proof of Theorem 2.2. $\square$

To prove Theorem 2.3, we first observe that in the chilled regime the Gibbs measure $\pi^G(\sigma)$ is such that

$$\pi^G(\sigma) = \frac{\mathrm{e}^{-2J\ell(\sigma)}}{1+o(1)}$$

where $o(1)$ denotes any function of L such that $\lim_{L\to\infty} o(1) = 0$. So if we let

$$\widehat{\pi}^G(\sigma) = \mathrm{e}^{-2J\ell(\sigma)},$$

we have clearly that

$$\lim_{L\to\infty} \frac{\pi^{(\leq 1)}(m)}{\pi^G(m)} = \lim_{L\to\infty} \frac{\pi^{(\leq 1)}(m)}{\widehat{\pi}^G(m)}. \tag{43}$$

We start computing $\widehat{\pi}^G(m)$. Observe that

$$\widehat{\pi}^G(m) = \mathrm{e}^{-4J} \sum_{i=1}^{L} \sum_{m=1}^{L-i} m + \sum_{m=1}^{L} m \sum_{k=2}^{L/2} \mathrm{e}^{-4kJ} n(k,m), \tag{44}$$

where $n(k,m)$ is the number of configurations with k disjoint intervals of minus spins with a total number m of minus spins. Due to the rough estimate $n(k,m) < L^{2k-1}$ we get

$$\widehat{\pi}^G(m) \leq \left[\frac{\mathrm{e}^{-4J}}{6}\left(L^3 - L\right) + L^3\mathrm{e}^{-4J}o(1)\right] \leq \frac{L^3\mathrm{e}^{-4J}}{6}(1+o(1)) \tag{45}$$

We next estimate the difference $\widehat{\pi}^G(m) - \pi^{(\leq 1)}(m)$. Observe that by (44)

$$\widehat{\pi}^G(m) \geq \mathrm{e}^{-4J} \sum_{i=1}^{L} \sum_{m=1}^{L-i} m$$

and that by (16) and (20)

$$\pi^{(\leq 1)}(m) = \pi^{(\leq 1)}_{(i;m)} = \mathrm{e}^{-4J} \pi^{(1)}_{(i;m)}$$

so we have

$$\widehat{\pi}^G(m) - \pi^{(\leq 1)}(m) \geq \mathrm{e}^{-4J} \sum_{i=1}^{L} \sum_{m=1}^{L-i} m(1 - \pi^{(1)}_{(i;m)})$$

Then note that, due to (39) we have that $1 - \pi^{(1)}_{(i;m)} > 0$, so we are allowed to restricted the sums over i, m above to a subset in which $i \leq m$. Recalling also bound (20) we get

$$\begin{aligned}\widehat{\pi}^G(m) - \pi^{(\leq 1)}(m) &\geq \mathrm{e}^{-4J} \sum_{i=1}^{L/2} \sum_{m=i}^{L-i} m(1 - \pi^{(1)}_{(i;m)}) \\ &\geq \mathrm{e}^{-4J} \sum_{i=1}^{L/2} \sum_{m=i}^{L-i} \left(m - 4\,\mathrm{e}^{-\frac{m^2}{2(i+m)}}\ m^2 \right) \\ &\geq \mathrm{e}^{-4J} \sum_{i=1}^{L/2} \sum_{m=i}^{L-i} \left(m - 4\,\mathrm{e}^{-\frac{m}{4}}\ m^2 \right) \geq \frac{\mathrm{e}^{-4J} L^3}{8} (1 + o(1))\end{aligned} \tag{46}$$

Hence, from inequalities (45) and (46), we get

$$\frac{\widehat{\pi}^G(m) - \pi^{(\leq 1)}(m)}{\widehat{\pi}^G(m)} \geq \frac{3}{4}\,(1 + o(1))$$

whence

$$\lim_{L \to \infty} \frac{\pi^{(\leq 1)}(m)}{\widehat{\pi}^G(m)} \leq \frac{1}{4}$$

and from (43) Theorem 2.3 immediately follows. □

5. Conclusions

In this paper, we have considered an example of a single spin-flip irreversible dynamics for a very simple system, but yet quite difficult to study in presence of boundary conditions. With explicit estimates we have shown that, expanding in series the stationary measure around the zero temperature, it is possible to control for very low temperature the convergence of the expansion and to compute, up to the first order, the stationary probability distribution. The latter has non-trivial features: It has an explicit dependence both on the relative distance and on the position of the changes of sign in the state. Moreover, the memory of the boundary conditions has a very slow decay and crucial macroscopic effects.

There are several questions opened by this result. The generalization of these computations to PCA dynamics, like the one presented in [4] and [5], should be straightforward. It should be possible also, with some extra effort, to understand the features of the higher terms of the expansion, and it would be very interesting to generalize this technique to higher dimensions. All these questions will be the subject of further investigations.

Acknowledgements

We thank Pietro Caputo for Eq. (37) and Francesco Pappalardi for useful discussions on the combinatorial properties of Catalan's triangles, and related references. We also thank A. De Sole, D. Gabrielli, G. Gallavotti, G. Jona-Lasinio for useful discussions. We also thank an anonymous referee for useful remarks. A.P. has been partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Fundação de Amparo à Pesquisa do Estado de Minas Gerais (FAPEMIG - Programa de Pesquisador Mineiro) and by the Simons Foundation and the Mathematisches Forschungsinstitut Oberwolfach. His stay in Rome during part of this work has been founded by the Grant "Visiting Professors" of the Universita' di Roma Tor Vergata. B.S. and E.S. thank the support of the A*MIDEX project (n. ANR-11-IDEX-0001-02) funded by the "Investissements d'Avenir" French Government program, managed by the French National Research Agency (ANR). B.S. has been supported by PRIN 2012, Problemi matematici in teoria cinetica ed applicazioni. E.S. has been supported by the PRIN 20155PAWZB "Large Scale Random Structures".

References

[1] Bayley, D.F.: Counting arrangements of 1's and -1's. Math. Mag. **69**, 128–131 (1996)

[2] Chou, T., Mallick, K., Zi, R.K.P.: Non-equilibrium statistical mechanics: from a paradigmatic model to biological transport. Rep. Prog. Phys. **74**, 116601 (2011)

[3] Costin, O., Lebowitz, J.L., Speer, E.R., Troiani, A.: The blockage problem. Bull. Inst. Math. Acad. Sin. N. S. **8**, 49–72 (2013)

[4] Dai Pra, P., Scoppola, B., Scoppola, E.: Sampling from a Gibbs measure with pair interaction by means of PCA. J. Stat. Phys. **149**, 722–737 (2012)

[5] Dai Pra, P., Scoppola, B., Scoppola, E.: Fast mixing for the low-temperature 2D Ising model through irreversible parallel dynamics. J. Stat. Phys. **159**, 1–20 (2015)

[6] De Carlo, L., Gabrielli, D.: Gibbsian stationary non equilibrium states. J. Stat. Phys. **168**, 1191–1222 (2017)

[7] Gallavotti, G.: Nonequilibrium and Irreversibility. Springer, Heidelberg (2014)

[8] Gaudillière, A., Landim, C.: A Dirichlet principle for non reversible Markov chains and some recurrence theorems. Probab. Theory Relat. Fields **158**, 55–89 (2013)

[9] Kolmogorov, A.: Zur theorie der Markoffschen ketten. Math. Ann. **112**, 155–160 (1936)

[10] Ng, S.A.: Some identities and formulas involving generalized Catalan numbers. arXiv:math/0609596w1 (2006)

[11] Procacci, A., Scoppola, B., Scoppola, E.: Probabilistic cellular automata for the low-temperature 2d Ising model. J. Stat. Phys. **165**, 991–1005 (2016)

[12] Robbins, H.: A remark on Stirlings formula. Am. Math. Mon. **62**, 26–29 (1955)

[13] Weisstein, E.W.: Catalan's triangle mathworld—a wolfram web resource. Retrieved 28 March 2012

Aldo Procacci
Departamento de Matemática
Universidade Federal de Minas Gerais
Av. António Carlos 6627
Belo Horizonte 30161-970
Brazil
e-mail: aldo@mat.ufmg.br

Benedetto Scoppola
Dipartimento di Matematica
Università di Roma"Tor Vergata"
Via della Ricerca Scientifica
00133 Rome
Italy
e-mail: scoppola@mat.uniroma2.it

Elisabetta Scoppola
Dipartimento di Matematica e Fisica
Università Roma Tre
Largo San Murialdo, 1
00146 Rome
Italy
e-mail: scoppola@mat.uniroma3.it

Communicated by Christian Maes.
Received: March 21, 2017.
Accepted: October 3, 2017.