Algebraic Geometry In Roma Tre

Graph complexes and moduli of Hassett-stable curves

Stefano Serpente,

Ph.D. Student, Università degli studi Roma Tre

GOAL: Find informations on the top weight cohomology of Hassett Moduli Spaces through combinatorial methods.

Background

Results

Tools and techniques

Hassett Moduli Spaces and Top Weight Cohomology

The introduction of the notion of stability allowed Deligne and Mumford to compactify the moduli space of smooth curves, and later with Knudsen they did the same for curves with marked points.
To generalize this fact, Hassett considered log-pairs (C, D) where C is a curve and D is an effective $\mathbb{Q}$-divisor of the form:

$$D = a_1p_1 + \ldots + a_np_n,$$

where $a_i \in (0,1] \cap \mathbb{Q}$, p_i are smooth points on the curve, and the notion of stability is the one coming from MMP, which for curves reduces to the following:

Definition (Hassett; '03)[1]:

Let $g \geq 0, n \geq 1$ be two integers, $\mathcal{A} = (a_1, \ldots, a_n) \in ((0,1] \cap \mathbb{Q})^n$ such that $2g - 2 + a_1 + \ldots + a_n > 0$.
Let C be a curve, $p_1, \ldots, p_n$ smooth points of C. Then C is $\mathcal{A}$-stable if
i) the sheaf $K_C + a_1p_1 + \ldots + a_np_n$ is ample;
ii) The points p_i for $i \in S \subseteq \{1, \ldots, n\}$ can coincide only if $\sum_{i \in S} a_i \leq 1$.

Theorem (Hassett; '03):

Let g, n and $\mathcal{A}$ as above. There exists a connected Deligne-Mumford Stack $\overline{\mathcal{M}}_{g, \mathcal{A}}$ smooth and proper over $\mathbb{Z}$ representing the moduli problem of curves of genus g which are $\mathcal{A}$-stable.

Remark:

When $\mathcal{A} = (1, \ldots, 1)$ is a sequence of n ones, Hassett stability coincides with Deligne-Mumford-Knudsen stability, and $\overline{\mathcal{M}}_{g, \mathcal{A}} = \overline{\mathcal{M}}_{g, n}$.

These $\overline{\mathcal{M}}_{g, \mathcal{A}}$'s behave analogously to the standard D.M.K. Moduli spaces. For example, we have the following generalization of a known property of $\overline{\mathcal{M}}_{g, n}$:

Theorem (Ulirsch; '15):

Consider the locus $\mathcal{M}_{g, \mathcal{A}} \subset \overline{\mathcal{M}}_{g, \mathcal{A}}$ of smooth curves. The boundary divisor

$$\partial \mathcal{M}_{g, \mathcal{A}} := \overline{\mathcal{M}}_{g, \mathcal{A}} \setminus \mathcal{M}_{g, \mathcal{A}}$$

is stack theoretically normal crossings.

Since $\mathcal{M}_{g, \mathcal{A}}$ and $\overline{\mathcal{M}}_{g, \mathcal{A}}$ are DM-stacks of dimension $3g - 3 + n$, by the work of Deligne we know that their rational cohomology admits a weight filtration

$$W = W_1 \subset \cdots \subset W_{6g-6+2n} \subseteq H^*(\mathcal{M}_{g, \mathcal{A}}, \mathbb{Q})$$

inducing a pure Hodge structure on the associated graded filtration.
Denote by $Gr^W_{6g-6+n} H^*(\mathcal{M}_{g, \mathcal{A}} \mathbb{Q}) := W_{6g-6+2n} / W_{6g-7+2n}$ the so called Top Weight Cohomology. We recall also the following result:

Theorem (Chan, Galatius, Payne; '18):

Let X be a smooth and separated DM-stack of dimension d with normal crossing compactification $\overline{X}$ and let $D = \overline{X} \setminus X$. Then there is a simplicial complex $\Delta(D)$ (constructed from the combinatorics of the intersections of the components of D)[2] with a natural isomorphism

$$Gr^W_{2d} H^{2d-k}(X; \mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta(D); \mathbb{Q}).$$

We call this Theorem ▲.

We want to apply the Theorem ▲ to $X = \mathcal{M}_{g, \mathcal{A}}$ and $\overline{X} = \overline{\mathcal{M}}_{g, \mathcal{A}}$, i.e. we want to make advantage of the isomorphism

$$Gr^W_{6g-6+n} H^{6g-6+n-k}(\mathcal{M}_{g, \mathcal{A}}; \mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta(\partial \mathcal{M}_{g, \mathcal{A}}); \mathbb{Q}).$$

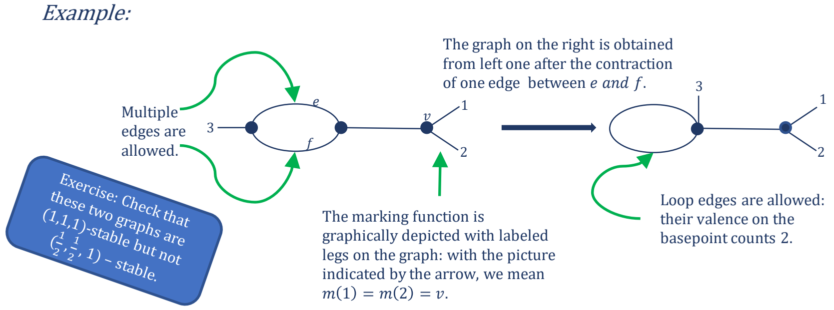
In particular, we want to compute the right hand side. As for $\overline{\mathcal{M}}_{g, n}$, each irreducible component of the boundary divisor of $\overline{\mathcal{M}}_{g, \mathcal{A}}$ has an associated dual graph, and these graphs can be arranged into chain complexes which compute $\widetilde{H}_{k-1}(\Delta(\partial \mathcal{M}_{g, \mathcal{A}}); \mathbb{Q})$.
Such complexes are usually called Graph complexes.

Graph Complexes

Definition[3]:

Fix $n \geq 1$ and $\mathcal{A} = (a_1, \ldots, a_n) \in ((0,1] \cap \mathbb{Q})^n$. A n-marked graph is a pair $\boldsymbol{G} = (G, m)$ where
- $G = (V(G), E(G))$ is a connected graph $(V(G) = \{vertices\ of\ G\}, E(G) = \{edges\ of\ G\})$;
- $m: \{1, \ldots, n\} \longrightarrow V(G)$ is called marking function;
The genus of the graph is $g(G) = |E(G)| - |V(G)| + 1$.
We say that the graph is $\mathcal{A}$-stable if for every vertex v of G, we have $val(v) - 2 + \sum_{i \in m^{-1}(v)} a_i > 1$.

Example:



For fixed g and $\mathcal{A}$, the set $\mathcal{G}_{g,\mathcal{A}}$ of $\mathcal{A}$-stable n-marked graphs of genus g is finite.
Consider the pairs $[\boldsymbol{G}, \omega]$, where $\boldsymbol{G}$ is in $\mathcal{G}_{g,\mathcal{A}}$ and ω is a total order on the set $E(G)$, with the relation $[\boldsymbol{G}, \omega] \sim \mathrm{sgn}(\sigma)[\boldsymbol{G}, \omega']$ if ω' is obtained from ω through the permutation σ.
We define *the graph complex* $\mathbb{G}^{(g,\mathcal{A})}$ to be the free $\mathbb{Q}$-vector space generated by the classes of this relation. The grading is given by declaring that $[\boldsymbol{G}, \omega]$ has degree $|E(G)| - 2$. It becomes a chain complex with the boundary map

$$\partial\, [\boldsymbol{G}, \omega] = \sum_{i=1}^{|E(G)|} (-1)^{i+1} [\boldsymbol{G}/e_i, \omega|_{E(G) \setminus \{e_i\}}],$$

Here the label of edges comes from ω. By $\omega|_{E(G) \setminus \{e_i\}}$ we mean the relative order induced by ω to the set of edges without e_i.

where if e_i is a loop we set the edge contraction $\boldsymbol{G}/e_i$ to be 0.

The first result we show is the following:

Theorem(Chan, Galatius, Payne;'19/ Kannan, Li, S., Yun '20)[4]:

$$H_k(\mathbb{G}^{(g,\mathcal{A})}) \cong \widetilde{H}_{k+2g-1}(\Delta(\partial \mathcal{M}_{g, \mathcal{A}}); \mathbb{Q}).$$

We call this isomorphism ★

Corollary:

$$Gr^W_{6g-6+n} H^{4g-6+n-k}(\mathcal{M}_{g, \mathcal{A}}; \mathbb{Q}) \cong H_k(\mathbb{G}^{(g,\mathcal{A})})$$

i.e. the top weight cohomology of $\mathcal{M}_{g, \mathcal{A}}$ is given by the homology of a graph complex.

Filtrations of $\mathbb{G}^{(g,\mathcal{A})}$

Now observe that given $\mathcal{A} = (a_1, \ldots, a_n), \mathcal{B} = (b_1, \ldots, b_n)$ so that $b_i \leq a_i$ for every i (if so we say $\mathcal{B} \leq \mathcal{A}$), if $\boldsymbol{G}$ is $\mathcal{B}$-stable then it is also $\mathcal{A}$-stable. This induces an injective map of chain complexes $\mathbb{G}^{(g,\mathcal{B})} \hookrightarrow \mathbb{G}^{(g,\mathcal{A})}$ sending a generator into itself.
So if we have a sequence $\mathcal{A}_1 \leq \mathcal{A}_2 \leq \ldots \leq \mathcal{A}_N$, we get a filtration

$$\mathbb{G}^{(g,\mathcal{A}_1)} \hookrightarrow \ldots \hookrightarrow \mathbb{G}^{(g,\mathcal{A}_N)}$$

This gives $\mathbb{G}^{(g,\mathcal{A}_N)}$ the structure of a filtered chain complex.
Given a filtered chain complex, we can then construct a spectral sequence $E^r_{p,q}$ which converges to its homology.

Theorem (S.; '21):

Given a sequence $\mathcal{A}_1 \leq \mathcal{A}_2 \leq \ldots \leq \mathcal{A}_N$, we have

$$Gr^W_{6g-6+n} H^{4g-6+n-k}(\mathcal{M}_{g,\mathcal{A}_N}; \mathbb{Q}) \cong \oplus_{i=1}^N E^\infty_{p,k-p},$$

where the terms $E^\infty_{p,k-p}$ are the ones to which the spectral sequence converges.

Proof sketch: By the theory of filtered chain complexes we have

$$H_k(\mathbb{G}^{(g,\mathcal{A}_N)}) \cong \underbrace{\oplus_{p+q=k} G_p H_{p+q}}_{\text{Filtered chain complex theory}} (\mathbb{G}^{(g,\mathcal{A}_N)}) \cong \underbrace{\oplus_{p=1}^N G_p H_k}_{\text{Bounded filtration}} (\mathbb{G}^{(g,\mathcal{A}_N)}) \cong \underbrace{\oplus_{i=1}^N E^\infty_{p,k-p}}_{\text{Convergence of the spectral sequence}}$$

Then we just apply ★ to get the result. ■

This theorem is useful because it allows us to have nonvanishing informations on $H^{4g-6+n-k}(\mathcal{M}_{g,\mathcal{A}_n}; \mathbb{Q})$, and for small g and n can be used to understand the whole Top Weight Cohomology:

Examples:

1) Through this filtration we were able to confirm a result of [CGP19], namely that

$$Gr^W_3 H^k(\mathcal{M}_{1,3}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & if\ k = 3 \\ 0 & o.w. \end{cases}$$

2) We are able to show the nonvanishing of $H^6(\mathcal{M}_{2,3}; \mathbb{Q})$ by computing one term of the direct sum.

Further directions and open problems

➤ Topology of $\Delta(\partial \mathcal{M}_{g,\mathcal{A}})$: through the identification with moduli spaces of tropical curves, lot of works has been done to understand the topology of this dual boundary complex in order to get informations on $\mathcal{M}_{g, \mathcal{A}}$. See for instance [CHMR14], [CMP+20], [KLSY20], [Yun21].

➤ Relative homology of the boundary complexes: It is possible to show that similar filtrations hold also for the dual boundary complexes. If $\mathcal{B} \leq \mathcal{A}$ there is an inclusion $\Delta(\partial \mathcal{M}_{g,\mathcal{B}}) \hookrightarrow \Delta(\partial \mathcal{M}_{g,\mathcal{A}})$. Then one can show that the relative homology of this inclusion is computed by $\mathbb{G}^{(g,\mathcal{A},\mathcal{B})} = \mathbb{G}^{(g,\mathcal{A})} / \mathbb{G}^{(g,\mathcal{B})}$ (see also [Ser21]).

➤ Describing $\mathbb{G}^{(g,\mathcal{A})}$ for big g and n: one issue to do more computations through this theorem is that it's hard to figure out all the generators of $\mathbb{G}^{(g,\mathcal{A})}$ as the parameters grow. It could be interesting to find a way to find them and/or compute kernels and images of the boundary map .

Footnotes

1. For experts: in the original work by Hassett, this definition is written more correctly for families of curves over a base scheme S. Here we just reduce ourself on a curve C for shortening purposes.
2. This space is identified with the moduli space of $\mathcal{A}$-stable tropical curves of genus g with volume 1. As we don't need too many details about it, we refer to [CGP21] and [KLSY20] for further explanations on it.
3. This definition is usually given with an extra function called weight function of the vertices, and a modified version of the genus. Since to define graph complexes we need only graphs where that function is identically 0 (as the others give only trivial classes), we avoid it for this dissertation.
4. The theorem was proven by the first group for the case $\mathcal{A} = (1, \ldots, 1)$, and then the second group showed that it works for any $\mathcal{A}$ without substantial changes in the proof. A proof for the general case is written in [Ser21], and follows the line of the original one in [CGP19]
5. The theorem on [Ser21] is stated after a deeper study of the wall crossing properties of the $\mathcal{A}$'s, this statement here is a reasonable reduction of the original one made for clarity purposes.

References:
[CGP19] Melody Chan, Søren Galatius, and Sam Payne. Topology of moduli spaces of tropical curves with marked points. arXiv e-prints, page arXiv:1903.07187, Mar 2019.
[CGP21] Melody Chan, Søren Galatius, and Sam Payne. Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$. Journal of the American Mathematical Society, 34(2):565–594, 2021.
[CHMR14] Renzo Cavalieri, Simon Hampe, Hannah Markwig, and Dhruv Ranganathan. Moduli spaces of rational weighted stable curves and tropical geometry. Forum of Mathematics, Sigma, 4, 04 2014.
[CMP+20] Alois Cerbu, Steffen Marcus, Luke Peilen, Dhruv Ranganathan, and Andrew Salmon. Topology of tropical moduli of weighted stable curves. Adv. Geom., 20(4):445–462, 2020.
[DM69] Pierre Deligne and David Mumford. The irreducibility of the space of curves of given genus. Publications Mathématiques de l'IH́ES, 36:75–109, 1969.
[Has03] Brendan Hassett. Moduli spaces of weighted pointed stable curves. Advances in Mathematics, 173(2):316 – 352, 2003.
[KLSY20] Siddarth Kannan, Shiyue Li, Stefano Serpente, and Claudia He Yun. Topology of tropical moduli spaces of weighted stable curves in higher genus. Adv. Geom., to appear
[Knu83] Finn F. Knudsen. The projectivity of the moduli space of stable curves, II: The stacks $\overline{\mathcal{M}}_{g, n}$. MATHEMATICA SCANDINAVICA, 52:161–199, Dec. 1983.
[Ser21] Stefano Serpente. Filtrations of moduli spaces of tropical weighted stable curves. arXiv preprint arXiv:2111.14965, 2021.
[Uli15] Martin Ulirsch. Tropical geometry of moduli spaces of weighted stable curves. Journal of the London Mathematical Society, 92(2):427–450, 2015.
[Yun21] Claudia He Yun. The S_n-Equivariant Rational Homology of the Tropical Moduli Spaces $\Delta_{2, n}$. Experimental Mathematics, pages 1–13, 2021.