Positivity of zero-regular bundles, continuous regularity and generic vanishing

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FIELDS

Introduction

We work over the field of complex numbers $\mathbb{C}$. We first define the CM–regularity.

Definition. Let X be a smooth projective variety and let H be an ample and globally generated line bundle on X. A coherent sheaf $\mathcal{F}$ on X is k*–regular* for $k \in \mathbb{Z}$ if $H^i(\mathcal{F}((k-i)H)) = 0$ for all $i \geq 1$.

Among the interesting classes of 0–regular bundles on X, one particular class, namely the class of Ulrich bundles, has been the focus of considerable activities in recent years. The main motivation for the first topic comes from the study of the positivity of Ulrich bundles that was initiated in [5]. In particular, the following theorem was established in [5] which shows that more generally, the positivity of $c_1(\mathcal{E})$ for a 0–regular bundle $\mathcal{E}$ depends on the projective geometry of X.

Theorem A. ([5] Theorem 7.2) *Let (X, H) be a smooth polarized projective variety of dimension n with H very ample. Also, let $\mathcal{E}$ be a 0–regular bundle of rank r for (X, tH) for some $t \geq 1$. Then for every $x \in X$ and for every subvariety $Z \subseteq X$ of dimension k passing through x, the following inequality holds*

$$c_1(\mathcal{E})^k \cdot Z \geq r^k mult_x(Z).$$

if either $t \geq 2$ or if $t = 1$ and x does not lie on a line contained in $\varphi_H : X \hookrightarrow \mathbb{P}^{h^0(H)-1}$.

Here we study the positivity when the polarization is divisible, i.e., when $t \geq 2$, and use it to study a continuous variant of CM–regularity introduced in [7]. Let us first recall some definitions.

Definition. Let X be a smooth projective variety and let $\mathcal{F}$ be a coherent sheaf on X. The *i–th cohomological support loci* $V^i(\mathcal{F})$ of $\mathcal{F}$ is defined as

$$V^i(\mathcal{F}) := \left\{ \zeta \in \mathrm{Pic}^0(X) \,|\, h^i(\mathcal{F} \otimes \zeta) \neq 0 \right\}.$$

Let H be an ample and globally generated line bundle on X. The sheaf $\mathcal{F}$ is called *continuously k–regular* for $k \in \mathbb{Z}$ if $V^i(\mathcal{F}((k-i)H)) \neq \phi$ for all $i \geq 1$.

Definition. Let X be a smooth projective variety and let $\mathcal{F}$ be a coherent sheaf on X. The sheaf $\mathcal{F}$ is called *Generic Vanishing* (abbreviated as *GV*) if $\mathrm{codim}(V^i(\mathcal{F})) \geq i$ for all integers $i > 0$. More generally, for an integer $k \geq 0$, $\mathcal{F}$ is called GV$_{-k}$ if codim$(V^i(\mathcal{F})) \geq i - k$ for all integers i.

We also recall the following definition for the future.

Definition. Let X be an abelian variety and let $\mathcal{F}$ be a coherent sheaf on X.
– $\mathcal{F}$ is called *Mukai regular* (abbreviated as *M–regular*) if $\mathrm{codim}(V^i(\mathcal{F})) > i$ for all $i > 0$.
– $\mathcal{F}$ is said to satisfy *Index Theorem with index* 0 (abbreviated as *IT*$_0$) if $V^i(\mathcal{F}) = \phi$ for all $i \neq 0$.

Using our study, we answer the following question of [7] in the affirmative for a class of polarizations on varieties whose Albanese maps are either surjective, or finite onto their images.

Question 1. ([7] Question $(*)$) Let X be a smooth projective variety of dimension $d \geq 1$ and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on X. Let $\mathcal{F}$ be a torsion–free coherent sheaf on X. If $\mathcal{F}$ is continuously 1–regular for $(X, \mathcal{O}_X(1))$, is $\mathcal{F}$ a GV sheaf?

The question above was motivated by the construction of rank 2 Ulrich bundles on abelian surfaces $(X, \mathcal{O}_X(1))$ in [2]. It turns out that for these bundles $\mathcal{E}$, $\mathcal{E}(-1)$ is indeed GV. The answer of Question 1 is affirmative for polarized curves, and it was shown in [7] that the answer of the question is also affirmative for many natural polarizations on well–known surfaces and certain ruled three–folds over curves.

Main results

We now define a partial variant of CM–regularity.

Definition. Let X be a smooth projective variety and let H be a globally generated line bundle on X. Also, let q, k be integers with $q \geq 0$. A coherent sheaf $\mathcal{F}$ on X is called $C_{q,k}$ for (X, H) if $H^{q+i}(\mathcal{F}((k-i)H)) = 0$ for all integers $i \geq 1$.

This partial variant was used earlier in [11], and the following result was established.

Lemma B. ([11] Lemma 3.2) *Let (X, H) be a smooth projective variety with H globally generated. Let $\mathcal{F}$ be a coherent sheaf on X, and q, k be integers with $q \geq 0$. If $\mathcal{F}$ is $C_{q,k}$ for (X, H) then it is $C_{q,k+m}$ for (X, H) for any integer $m \geq 0$.*

As a consequence of Lemma B, we deduce the following proposition.

Proposition C. *Let X be a smooth projective variety and let H_1 and H_2 be globally generated line bundles on X. Let $t_1, t_2 \geq 1$ be integers and let $\mathcal{F}$ be a coherent sheaf on X that is $C_{0,0}$ for $(X, t_1H_1 + t_2H_2)$. Then, for any $q \in \mathbb{N}$ with $1 \leq q \leq \dim X$, $\mathcal{F}(-qt_1H_1 - (qt_2 - 1)H_2)$ is $C_{q-1,0}$ for (X, H_2).*

Definition. Let $\mathcal{E}$ be a vector bundle on a smooth projective variety X. The bundle $\mathcal{E}$ is called *k–jet ample* if for every choice of t distinct points $x_1, \cdots, x_t \in X$ and for every tuple $(k_1, \cdots, k_t)$ of positive integers with $\sum k_i = k + 1$, the following evaluation map surjects

$$H^0(\mathcal{E}) \to H^0\left(\mathcal{E} \otimes \left(\mathcal{O}_X/(\mathfrak{m}_{x_1}^{k_1} \otimes \cdots \otimes \mathfrak{m}_{x_t}^{k_t})\right)\right) = \bigoplus_{i=1}^{t} H^0\left(\mathcal{E} \otimes \left(\mathcal{O}_X/\mathfrak{m}_{x_i}^{k_i}\right)\right).$$

As a consequence of Theorem A, Proposition C, and results of [1], we deduce

Corollary D. *Let X be a smooth projective variety and let H be an ample and globally generated line bundle on X. Also, let $\mathcal{E}$ be a 0–regular vector bundle for (X, tH) of rank r for some $t \in \mathbb{N}$, $t \geq 2$. Then $\mathcal{E}(-(t-1)H)$ is 0–regular for (X, H); in particular it is globally generated. Moreover, if H is very ample, then*
(1) the vector bundle $\mathcal{E}$ is $(t-1)$–jet ample;
(2) for every $x \in X$ and for every subvariety $Z \subseteq X$ of dimension k passing through x, the following holds if x does not lie on a line contained in $\varphi_H : X \hookrightarrow \mathbb{P}^{h^0(H)-1}$

$$c_1(\mathcal{E})^k \cdot Z \geq r^k mult_x(Z) + r^k(t-1)^k H^k Z.$$

Now we state our first main result towards generic vanishing.

Theorem E. *Let (X, H) be a polarized smooth projective variety with H globally generated, and assume that $alb_X : X \to Alb(X)$ is surjective. Further, assume that there exists a globally generated line bundle H_1 on X and an ample line bundle H_2 on $Alb(X)$ such that $H = H_1 + alb_X^* H_2$. Let $\mathcal{F}$ be a torsion–free coherent sheaf on X that is continuously 1–regular for (X, H). Then $\mathcal{F}$ is a GV sheaf.*

We further consider the following more general question.

Question 2. Let X be a smooth projective variety of dimension $d \geq 1$ and let $\mathcal{O}_X(1)$ be an ample and globally generated line bundle on X. Let $\mathcal{F}$ be a torsion–free coherent sheaf on X. If $\mathcal{F}$ is continuously k–regular for $(X, \mathcal{O}_X(1))$ for some integer $1 \leq k \leq d$, is $\mathcal{F}$ a GV$_{-(k-1)}$ sheaf?

The main evidence for the validity of the above is the following

Theorem F. *Let X be a smooth projective variety of dimension d and assume that its Albanese map $alb_X : X \to Alb(X)$ is finite onto its image. Let H_2 be an ample line bundle on X and let $H := H_1 + alb_X^* H_2$ where H_1 is globally generated. Assume H is also globally generated, and let $\mathcal{F}$ be a continuously k–regular torsion–free coherent sheaf for (X, H) for some integer $1 \leq k \leq d$. Then the sheaf $\mathcal{F}$ is GV$_{-(k-1)}$.*

The arguments of Theorem E and Theorem F are similar. Assume in Theorem F, we have $k = 1$ and $H_1 = \mathcal{O}_X$. We first generalize the definition of continuous CM–regularity to a relative set–up. Given a morphism $a : X \to A$ to an abelian variety A, we call a sheaf $\mathcal{F}$ *a–continuously k–regular* if the relative cohomological support loci $V_a^i(\mathcal{F}((k-i)H) \neq \phi$ for all $i \geq 1$. To this end, we use a covering trick used by Pardini in [8]. More precisely, we set $A := \mathrm{Alb}(X)$, $a := \mathrm{alb}_X$, and consider the following base–change diagram

$$\begin{array}{ccc} X_n & \xrightarrow{\mu_n} & X \\ \downarrow & & \downarrow \\ A & \xrightarrow{n_A} & A \end{array}$$

where n_A is the multiplication map by n. Let $a_n : X_n \to A$ be the left vertical map. Since $\mathcal{F}(H)$ is continuously 0–regular for (X, H) by assumption, it turns out that there exists $\zeta \in \mathrm{Pic}^0(A)$ such that $\mu_n^*\mathcal{F} \otimes a_n^*(n^2H_2 + \zeta)$ is 0–regular (in the usual sense) for $(X_n, n^2a_n^*H_2)$. Using Proposition C, we now deduce that when n is 2–divisible,

$$\mu_n^*\mathcal{F} \otimes \left(a_n^*(n^2H_2 + \zeta) - \left(\frac{n^2}{2} - 1\right) a_n^*(2H_2) \right) = \mu_n^*\mathcal{F} \otimes a_n^*(2H_2 + \zeta)$$

is 0–regular for $(X_n, a_n^*(2H_2))$. The key property of $a_n^*(2H_2)$ is that for any $\zeta' \in \mathrm{Pic}^0(A)$, $a_n^*(2H_2 + \zeta')$ is globally generated. We now prove the following result.

Proposition G. *Let (X, H) be a polarized smooth projective variety and let $a : X \to A$ be a morphism to an abelian variety A. Assume (X, H) satisfies the property that $H + a^*\zeta$ is globally generated for all $\zeta \in Pic^0(A)$. Let $\mathcal{F}$ be a torsion–free coherent sheaf on X that is a–continuously 0–regular for (X, H). Then $V_a^i(\mathcal{F}) = \phi$ for all integers $i \geq 1$.*

Continuing the proof, the above implies $V_{a_n}^i(\mu_n^*\mathcal{F} \otimes a_n^*(2H_2)) = \phi$ for all $i \geq 1$. In other words, $a_{n*}(\mu_n^*\mathcal{F}) \otimes (2H_2)$ is IT$_0$. Thus, for any ample line bundle L on A, for $n \gg 0$, $n^2L - 2H_2$ will be ample, and consequently by the preservation of vanishing in [9], we deduce that $a_{n*}(\mu_n^*(\mathcal{F} \otimes a^*L))$ is IT$_0$, whence $H^i(\mu_n^*(\mathcal{F} \otimes a^*L)) = 0$ for all $i \geq 1$. This in turn implies by the projection formula that $H^i(\mathcal{F} \otimes a^*L) = 0$ for all $i \geq 1$. Similar argument using a criterion of generic vanishing via Fourier–Mukai transforms ([6]) proven in [4], [10] proves Theorem F for $k = 1$ and $H_1 = \mathcal{O}_X$.

Using our results, and theorems in [9], [3], we deduce the following

Corollary H. *Let (X, H) be a polarized abelian variety with H globally generated, and let $\mathcal{F}$ be torsion–free coherent sheaf on X. Assume $\mathcal{F}$ is continuously k–regular for (X, H) for some $k \in \mathbb{N}$ with $1 \leq k \leq \dim X$. Then $\mathcal{F}$ is GV$_{-(k-1)}$. In particular, (1) if $k = 1$, then $\mathcal{F}$ is nef; (2) if $k = 0$, then $\mathcal{F}$ is ample.*

Definition. Let X be a smooth projective variety. A very ample line bundle L is said to satisfy property N_0 if $|L|$ embeds X as a projectively normal variety. A very ample line bundle L satisfies property N_1 if L satisfies property N_0 and the homogeneous ideal I of the image of X embedded by $|L|$ is generated by quadratic equations. Finally a very ample line bundle L is said to satisfy property N_p, $p \geq 1$, if it satisfies property N_1 and the matrices in the minimal graded free resolution of S/I have linear entries from the second to the p–th step.

Corollary I. *Let (X, H) be a polarized abelian variety with H globally generated. Also let $\mathcal{E}$ be a 0–regular vector bundle for (X, tH). Then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ satisfies N_p property if $t \geq p + 2$.*

Definition. A line bundle L on X is called *N–Koszul* for a natural number N, if the sections of L give a projective embedding of X, and the homogeneous coordinate ring $A = \bigoplus_{q \geq 0} H^0(qL)$ is N-Koszul, i.e., $\mathbb{C}$ has a resolution as a graded A–module, $\cdots \to M_1 \to M_0 \to \mathbb{C} \to 0$, with M_i a free module generated in degree i for $i \leq N$.

Using Proposition C, and a resolution of diagonal constructed in [11], we prove

Corollary J. *Let X be a smooth projective variety of dimension n with a very ample line bundle H. Assume for some $p \geq 0$, $t \geq p + 1$ and $R(kH)$ is $3n$–Koszul for all $1 \leq k \leq p + 1$. Let $\mathcal{E}$ be an ample and 0–regular vector bundle for (X, tH). Then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ satisfies N_p property.*

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