CAYLEY-BACHARACH PROPERTY AND APPLICATIONS

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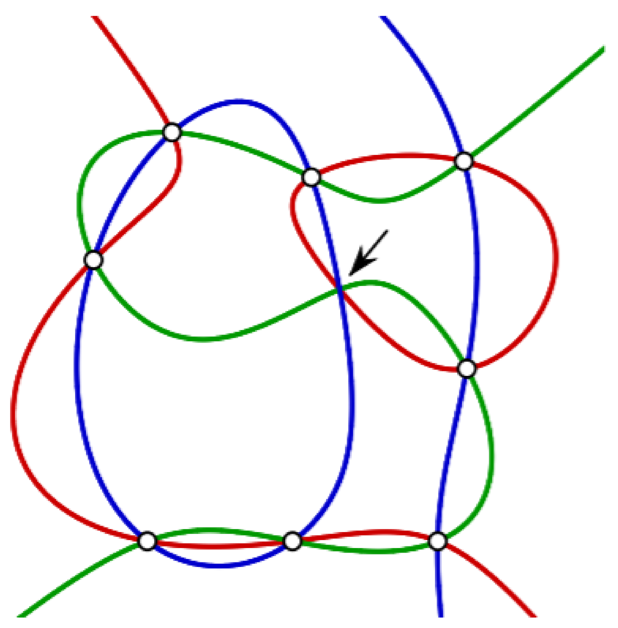
Motivations

The celebrated Cayley-Bacharach theorem asserts that if

$$\Gamma = \{P_1, \dots, P_{de}\} \subset \mathbb{P}^2$$

is the collection of distinct intersection points between two plane curves of degree d and e respectively, then any curve of degree $d+e-3$ passing through all but one points of Γ passes through the last point.



All cubics passing through the eight white points meet in a unique ninth point

This is a basic version of a very classical property that come up in many field of modern algebraic geometry. For instance, it is useful for studying linear series on curves. Moreover, in the last decade there has been a new and growing interest for this property due to its applications to the study of *measures of irrationality* for projective varieties, i.e. birational invariants that measure how a variety is far from satisfying properties that are distinctive of the projective space.

Cayley-Bacharach property

Definition. *A set of distinct points*

$$\Gamma = \{P_1, \dots, P_d\} \subset \mathbb{P}^n$$

satisfies the ***Cayley-Bacharach condition*** *with respect to the complete liner system $|\mathcal{O}_{\mathbb{P}^n}(k)|$ of hypersurfaces of degree k, or more briefly Γ is $CB(k)$, if for every $i = 1, \dots, d$ and for any effective divisor $D \in |\mathcal{O}_{\mathbb{P}^n}(k)|$ passing through $P_1, \dots, \widehat{P_i}, \dots, P_d$, we have $P_i \in D$ as well.*

Main References

[1] F. Bastianelli, On the symmetric products of curves, *Trans. Amer. Math. Soc.*, **364**(5) (2012), 2493–2519.

[2] F. Bastianelli, R. Cortini and P. De Poi, The gonality theorem of Noether for hypersurfaces, *J. Algebraic Geom.*, **23**(2) (2014), 313–339.

[3] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, and B. Ullery, Measures of irrationality for hypersurfaces of large degree, *Compos. Math.*, **153** (2017), no. 11, 2368–2393.

[4] A. F. Lopez and G. P. Pirola, On the curves through a general point of smooth surface in $\mathbb{P}^3$, *Math. Z.*, **219** (1994), 93–106.

[5] N. Picoco, Geometry of points satisfying Cayley-Bacharach conditions and applications, *arXiv:2201.01665v2* (2022).

Geometry of points $CB(k)$ in the projective space

Following the approach of Lopez and Pirola in [**4**], we prove

Theorem A

Let $\Gamma = \{P_1, \dots, P_d\} \subset \mathbb{P}^n$ be a set of distinct points satisfying the Cayley-Bacharach condition with respect to $|\mathcal{O}_{\mathbb{P}^n}(k)|$, with $k \geq 1$. For any $3 \leq h \leq 5$, if

$$d \leq h(k-h+3) - 1$$

then Γ lies on a curve of degree $h-1$.

We point out that

- the theorem is true also for $h = 2$ (see [**2**, **Lemma 2.4**]);
- cases $h = 3, 4, 5$ in $\mathbb{P}^2$ and cases $h = 3, 4$ in $\mathbb{P}^3$ are covered by [**4**, **Lemma 2.5**].

The cases $h = 3, 4$ are achieved by induction on the dimension of the ambient space. On the other hand, the case $h = 5$ is more complicated, because the induction argument requires to prove separately the cases $n = 3$ and $n = 4$. To this aim we extend to this setting the argument of [**4**]. In particular, we first prove that Γ lies on a reduced curve C of degree at most 9. Then we distinguish several cases (depending on the irreducible components of C) showing that the sum of the degrees of the irreducible components C_i of C such that $C_i \cap \Gamma \neq \emptyset$ is at most 4, as wanted.

Applications

As first application we extend [**4**, **Theorem 1.5**] about **linear series on curve** in projective 3-space .

Theorem 1. *Let $S \subset \mathbb{P}^3$ be a smooth surface of degree $d \geq 5$, let C be an integral curve on S such that $|\mathcal{O}_C \otimes \mathcal{O}_S(C)|$ is base point free and let L be a base point free special g^r_n on C that is not composed with an involution if $r \geq 2$.*
If $n \leq 5d - 31$, there exists an integer h, with $1 \leq h \leq 4$, such that

$$h(d-h-1) \leq n \leq \min\{hd, (h+1)(d-h-2) - 1\}$$

and the general divisor of the g^r_n lies on a curve of degree h.

Our contribution is the improvement of the upper bound on n (which was $n \leq 4d - 21$ in [**4**, **Theorem 1.5**]).

The second application concerns the so-called **correspondences with null trace**.
Let X, Y be two projective varieties of dimension n, with X smooth and Y integral. A *correspondence of degree d* on $Y \times X$ is an integral n-dimensional variety $\Sigma \subset Y \times X$ such that the projections $\pi_1 : \Sigma \to Y, \pi_2 : \Sigma \to X$ are generically finite dominant morphisms and $\deg \pi_1 = d$. Let $U \subset Y_{\text{reg}}$ be an open subset such that $\dim \pi_1^{-1}(y) = 0$ for every $y \in U$. Associate to Σ there is a map $\gamma : U \to X^{(d)}$, defined by

$$\gamma(y) = P_1 + \cdots + P_d,$$

where $\pi_1^{-1}(y) = \{(y, P_i) | i = 1, \dots, d\}$.
Linked to the map γ it is possible to define the *Mumford's trace map* (see e.g. [**4**, **Section 2**])

$$Tr_\gamma : H^{n,0}(X) \to H^{n,0}(U).$$

We say that Σ is a *correspondence with null trace* if $Tr_\gamma = 0$.

Theorem 2. *Let $n \geq 3$ and let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree $d \geq n+2$. Let Σ a correspondence of degree m with null trace on $Y \times X$.*
If $m \leq 5(d-n) - 16$, the only possible values of m are

- $d - n + 1 \leq m \leq d,$
- $2(d-n) \leq m \leq 2d,$
- $3(d-n-1) \leq m \leq 3d,$
- $4(d-n-2) \leq m \leq 4d.$

The proof of this theorem relies on the argument of [**4**, **Theorem 1.3**], which holds in $\mathbb{P}^3$, and extends it to any $\mathbb{P}^n$ with $n \geq 3$.

Cayley-Bacharach property on Grassmannians

Let $\mathbb{G}(k-1, n)$ be the Grassmann variety of $(k-1)$-planes in $\mathbb{P}^n$ and let $\Gamma = \{P_1, \dots, P_d\}$ be a set of points in $\mathbb{G}(k-1, n)$ satisfying the Cayley-Bacharach condition with respect to the complete linear series $|\mathcal{O}_{\mathbb{G}(k-1,n)}(1)|$. For any $(n-k)$-plane $L \subset \mathbb{P}^n$, the *Schubert cycle*

$$\sigma_1(L) := \Big\{[\Lambda] \in \mathbb{G}(k-1, n) | \Lambda \cap L \neq \emptyset\Big\}$$

is an effective divisor of $|\mathcal{O}_{\mathbb{G}(k-1,n)}(1)|$. This leads to

Definition. *Let $\Lambda_1, \dots, \Lambda_d \subset \mathbb{P}^n$ be $(k-1)$-planes. We say that they are in* ***special position*** *with respect to $(n-k)$-planes, or briefly that they are $SP(n-k)$, if for every $i = 1, \dots, d$ and for any $(n-k)$-plane $L \subset \mathbb{P}^n$ intersecting $\Lambda_1, \dots, \widehat{\Lambda_i}, \dots, \Lambda_d$, we have $\Lambda_i \cap L \neq \emptyset$, too.*

Theorem B (joint with F. Bastianelli)

Let $\Lambda_1, \dots, \Lambda_d \subset \mathbb{P}^n$ be $(k-1)$-planes $SP(n-k)$. Assume, moreover, that there exists no partition of $\{\Lambda_1, \dots, \Lambda_d\}$ such that any part is $SP(n-k)$ itself. Then

$$\dim \operatorname{Span}(\Lambda_1, \dots, \Lambda_d) \leq d + k - 3.$$

Application to measures of irrationality on $C^{(k)}$

One of the main extensions of the notion of gonality to higher dimensional varieties is the **covering gonality** (cf. [**3**])

$$\operatorname{cov.gon}(X) := \min\left\{ d \in \mathbb{N} \;\middle|\; \begin{array}{c} \text{given a general point } x \in X, \\ \exists \text{ an irreducible curve } C \subset X \\ \text{through } x \text{ with } \operatorname{gon}(C) = d \end{array} \right\}$$

In [**1**] Bastianelli proved that if C is a smooth curve of genus $g \geq 3$, then $\operatorname{cov.gon}(C^{(2)}) = \operatorname{gon}(C)$. We prove the same for 3-fold and 4-fold symmetric products of curves. Namely,

Theorem 3 (joint with F. Bastianelli). *Let $k \in \{3, 4\}$ and let C be a smooth complex non-hyperelliptic projective curve of genus $g \geq k+1$. Then*

$$cov.gon(C^{(k)}) = gon(C)$$

unless $(k, g, gon(C)) = (4, 5, 4)$.

As for any $k \geq 2$ the variety $C^{(k)}$ is covered by the family $\{C_P\}_{P \in C^{(k-1)}}$ of curves $C_P := C + P = \{q + P | q \in C\}$, it is immediate to see that $\operatorname{cov.gon}(C^{(k)}) \leq \operatorname{gon}(C)$. Moreover, as $C^{(k)}$ is not covered by rational curves, it follows that the assertion holds when $\operatorname{gon}(C) = 2$.

The hardest part is to prove that $\operatorname{cov.gon}(C^{(k)}) \geq \operatorname{gon}(C)$ when C is non-hyperelliptic. To this aim we consider the canonical model of $C \subset \mathbb{P}^{g-1}$ and a family $\mathcal{E} = \{E_t\}_{t \in T}$ of d-gonal curves covering $C^{(k)}$, where T is a variety of dimension $k-1$.
For general $t \in T$, let $\nu \colon \tilde{E}_t \to E_t$ be the normalization map and let $f_t \colon \tilde{E}_t \to \mathbb{P}^1$ be a map of degree d. Moreover, let $\{\tilde{P}_1, \dots, \tilde{P}_d\}$ be the general fiber of f_t. Finally, for $i = 1, \dots, d$, let $P_i := \nu(\tilde{P}_1)$ with

$$P_i = p_{i_1} + \cdots + p_{i_k} \in C^{(k)}$$

and let

$$\Lambda_i := \operatorname{Span}(p_{i_1}, \dots, p_{i_k}) \subset \mathbb{P}^{g-1}.$$

It turns out that the $(k-1)$-planes Λ_i are $SP(n-k)$.
Let us consider the effective divisor

$$D := p_{1_1} + \cdots + p_{d_k}.$$

By Theorem B, we can bound from above the dimension of

$$\operatorname{Span}\Big(\Lambda_1, \dots, \Lambda_d\Big) = \operatorname{Span}\Big(p_{1_1}, \dots, p_{d_k}\Big).$$

Then, by the geometric version of Riemann-Roch theorem, we get a lower bound for the dimension of the linear series on C given by $|D|$. Distinguishing the cases in which the points p_{i_j} are or not distinct, we get $d \geq \operatorname{gon}(C)$.

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