Donaldson-Thomas theory using finite and local fields

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Introduction

This poster is about the computation of Donaldson-Thomas invariants. They are invariants of **moduli spaces of sheaves** over a smooth projective variety of low dimension (up to 4). For example, such enumerative invariants are sensible ways of countings curves in surfaces or 3-folds. Although they were defined first as integrals over a virtual class, the development of a motivic Donaldson-Thomas theory leads us to methods such as finite fields counting and p-adic integration (the latter was suggested to us by Francesca Carocci).

Donaldson-Thomas invariants

Let X be a smooth surface or 3-fold (over $\mathbb{C}$). We would like to count the number of sheaves of some fixed topological data over X. Let $\mathcal{M}$ be a moduli of sheaves with fixed topological data ν. They often carries a perfect obstruction theory hence equipped with a virtual class $[\mathcal{M}]^{vir}$ by Behrend-Fantechi. It is a 0-cycle if X is Calabi-Yau and $\mathcal{M}$ is the Hilbert scheme of curves of fixed topological data ν. The **Donaldson-Thomas invariants** are just :

$$DT_\nu(X) = \int_{[\mathcal{M}^{vir}]} 1$$

Surprisingly, these numbers can be recovered independently as a weighted-Euler characteristic

$$DT_\nu(X) = \chi_{vir}(\mathcal{M})$$

whenever the tangent obstruction theory is symmetric. The weight is given by a constructible **Behrend function** ν_B. In cases where the moduli is locally written as a critical locus, there is a description of ν_B in terms of Milnor fibres. It leads to the definition of a **motivic Donaldson-Thomas invariant**, a virtual motive $[\mathcal{M}]^{mot}$ which satisfies $\chi([\mathcal{M}]^{mot}) = \chi_{vir}(\mathcal{M})$.

P-adic integration

Let $x \in \mathbb{Q}$. The p-adic norm of x is given by $\frac{1}{p^m}$, where $m \in \mathbb{Z}$ is the order of x. The p-adic field $\mathbb{Q}_p$ is the Cauchy completion of $\mathbb{Q}$ for this norm. For such locally compact abelian groups, there is a translation-invariant measure μ called **Haar measure**. We can normalise it such that $\mu(\mathbb{Z}_p) = 1$, where $\mathbb{Z}_p$ is the ring of p-adic integers. From that data, we can compute the measure of any constructible set using translation invariance. For example, $\mathbb{Z}_p = p\mathbb{Z}_p \sqcup (p\mathbb{Z}_p + 1) \sqcup \cdots \sqcup (p\mathbb{Z}_p + (p-1))$ gives $\mu(p\mathbb{Z}_p) = \frac{1}{p}$.

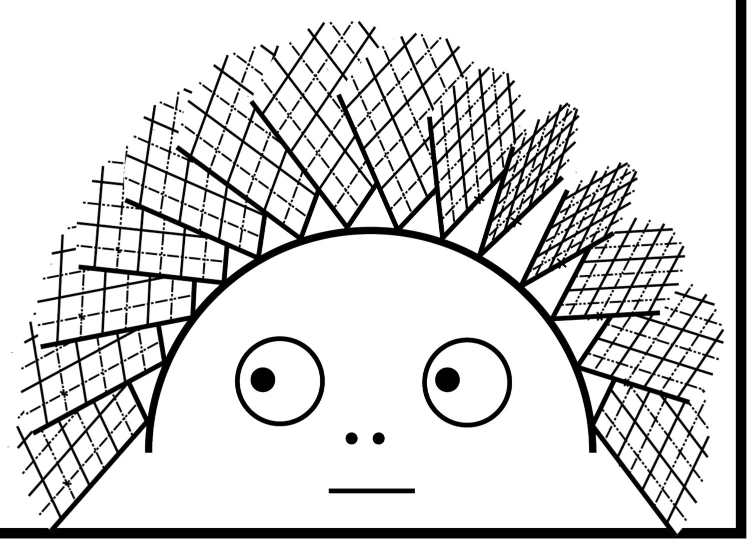
Example 1 *Let $f : \mathbb{Z}_p \to \mathbb{Z}_p$ be defined by $f(x) = x^n$. We compute $Z(f,s) = \int_{\mathbb{Z}_p} |f(x)|^s d\mu$. If $x \in p^m(\mathbb{Z}^p \setminus p\mathbb{Z}_p)$, $|x^n| = \frac{1}{p^{mn}}$. We get*

$$\begin{aligned}\int_{\mathbb{Z}_p} |x^n|^s d\mu &= \mu(\mathbb{Z}_p \setminus p\mathbb{Z}_p) \times 1 + \mu(p\mathbb{Z}_p \setminus p^2\mathbb{Z}_p) \times \tfrac{1}{p^{ns}} + \ldots \\ &= (1 - \tfrac{1}{p}) \times 1 + (\tfrac{1}{p} - \tfrac{1}{p^2}) \times \tfrac{1}{p^{ns}} + \cdots = \tfrac{p-1}{p-p^{-ns}}\end{aligned}$$

To define Donaldson-Thomas types invariants via p-adic integration, we need to generalise the set-up to K**-analytic manifold**, where K is a p-adic field (e.g. $K = \mathbb{Q}_p$). It can be done locally by using a gauge form ω on $\mathcal{M}(K)$ and its associated Weil measure and globally by gluing these measures into a canonical measure μ_{can}.

A toy example : the resolved conifold

A natural example to test computations of 1-dimensional DT invariants is the **resolved conifold**, i.e. the total space of the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbb{P}^1$. The only compact curve in the resolved conifold is $\mathbb{P}^1$ because $\mathcal{O}(-1)$ has no sections. The topological support of sheaves in our moduli is fixed as $\beta = [\mathbb{P}^1]$.
Numerical DT invariants are known, as well as motivic DT invariants.
Doing the point counting on $\mathcal{M}(\mathbb{F}_q)$ in that case only works for $\chi = 1, 2$ because for $\chi > 2$ the moduli is singular. These motivic DT invariants have been computed in [3] using a quiver description of the conifold.
The resolved conifold can be obtained by a resolution of singularity. This singularity can be associated with a **non-commutative quiver algebra**. Hilbert scheme of 1-dimensional subschemes can be expressed as a **Crit**(f) of a function $f : S \to \mathbb{C}$ where S is a smooth variety.



Point countings on $\mathcal{M}(\mathbb{F}_q)$

Another way to count sheaves is via **counting** $\mathbb{F}_q$**-points** $\#\mathcal{M}(\mathbb{F}_q)$. This approach has been historically developed with Weil's conjecture and is often fruitful whenever one would like to compute a motivic invariant.

Euler characteristic is the most natural motivic invariant. For example, $\#\mathbb{P}^n_{\mathbb{F}_q} = q^n + q^{n-1} + \cdots + 1$. Substituting $q = 1$ gives $\chi(\mathbb{P}^n_{\mathbb{C}}) = n + 1$, and $q = -1$ gives $\chi(\mathbb{P}^n_{\mathbb{R}})$

Can we simply count the $\mathbb{F}_q$-points on our moduli ?
This works when the moduli is **smooth**, because in that case the Behrend function is just ± 1, so DT invariants are just signed Euler characteristic. It requires some knowledge about the moduli (e.g. equations over $\mathbb{Z}$ of its strata). An example to test it is the moduli of one-dimensional sheaves of low degree on $\mathbb{P}^2$. They are not rational and have been studied by Le Potier, and later on Choi and Chung (degree 4,5).

References

[1] Kai Behrend, Jim Bryan, and Balazs Szendroi. *Motivic degree zero Donaldson-Thomas invariants*. 2009. URL: `https://arxiv.org/abs/0909.5088`.
[2] Francesca Carocci, Giulio Orecchia, and Dimitri Wyss. *BPS invariants from p-adic integrals*. 2021. URL: `https://arxiv.org/abs/2112.12103`.
[3] Andrew Morrison, Sergey Mozgovoy, Kentaro Nagao, and Balazs Szendroi. *Motivic Donaldson-Thomas invariants of the conifold and the refined topological vertex*. 2011. URL: `https://arxiv.org/abs/1107.5017`.

Towards p-adic DT invariants

We can compare Example 1 with **motivic computations**. $f(x) = x^n$ (over $\mathbb{C}$) has **Crit**(f) = **Spec**$\mathbb{C}[x]/(nx^{n-1})$ and can be associated with a motivic zeta function $Z_f(T)$ which is rational and reads :

$$Z_f(T) = [E \mapsto U_0, \rho_I] \frac{\mathbb{L}^{-\nu_0} T^{N_0}}{1 - \mathbb{L}^{-\nu_0} T^{N_0}}$$

where $[E \mapsto U_0, \rho_I]$ is some element of the monodromic Grothendieck group called motivic nearby cycle. In our case $\nu_0 = 1$ and $N_O = n$ and we recover the local zeta function of the p-adic context, the only difference being the nearby cycle.

$\mathcal{M}$ is a singular moduli embedded in a smooth projective variety as a critical locus of $f : S \to \mathbb{C}$ for any moduli of stable sheaves over a Calabi-Yau 3-fold X. In this case, if f can be expressed as a K-analytic function over $S(K)$ for K p-adic field, we can study

$$pDT = \int_{S(\mathcal{O}_K)} |f(x)|^s - 1 \, d\mu_{can}$$

Can we recover motivic computations done for Hilbert schemes of points case ([1]) and for the conifold case ? What is the good notion of limit on s ?
When a moduli contains strictly semi-stable sheaves, the **pBPS invariants** defined in [2] are defined as integrals of a gerbe function from the corresponding stack to the coarse moduli.