Complete intersections subvarietes of Veronese surfaces

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Complete intersections on Veronese surfaces

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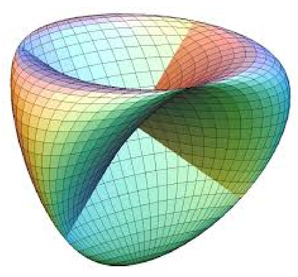
Question

Which are the complete intersections lying on Veronese surfaces?

General setting

Let $\nu_{n,d}$ be the Veronese map:

$$\nu_{n,d} : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{n}-1}, \qquad V_{n,d} = \nu_{n,d}(\mathbb{P}^n)$$



The (singular) embedding of $V_{2,2}$ in $\mathbb{P}^3$: the Steiner surface

We showed that, if $\mathbb{X} \subseteq V_{n,d}$ is a subvariety and we set $\mathbb{Y} = \nu_{n,d}^{-1}(\mathbb{X})$, then it holds that $(\mathcal{I}(\mathbb{X}))_t = (\mathcal{I}(\mathbb{Y}))_{dt}$. Hence we have

$$H_{\mathbb{X}}(t) = H_{\mathbb{Y}}(dt)$$

and that allows us to prove the following theorem:

Hilbert functions of subvarieties of $V_{n,d}$

Let $h(t) : \mathbb{N} \to \mathbb{N}$ be the Hilbert function of a projective variety in $\mathbb{P}^N$. Then there exists $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^N$ such that $H_{\mathbb{X}}(t) = h(t)$ if and only there exists $k(t) : \mathbb{N} \to \mathbb{N}$ Hilbert function of a projective variety in $\mathbb{P}^n$ such that $h(t) = k(dt)$.

d-sequences

Inspired by the definition of differentiable 0-sequences, given by A.V. Geramita, P. Maroscia and L.G. Roberts in 1983 to characterize Hilbert functions of reduced varieties, we define differentiable d-sequences as follows:

- A sequence of non-negative integers $(c_t)_{t\in\mathbb{N}}$ is called a 0-*sequence* if $c_0 = 1$ and $c_{t+1} \leq c^{\langle t \rangle}$ for all $t \geq 1$.
- Let $(b_t)_{t\in\mathbb{N}}$ be a 0-sequence. Then $(b_t)_{t\in\mathbb{N}}$ is *differentiable* if the difference sequence $(c_t)_{t\in\mathbb{N}}, c_t = b_t - b_{t-1}$ is again a 0-sequence (where $b_{-1} = 0$).
- A 0-sequence $(b_t)_{t\in\mathbb{N}}$ is called d-*sequence* if there exists a 0-sequence $(c_t)_{t\in\mathbb{N}}$ such that $b_t = c_{(d+1)t}$.
- A 0-sequence $(b_t)_{t\in\mathbb{N}}$ is called *differentiable d-sequence* if there exists a differentiable 0-sequence $(c_t)_{t\in\mathbb{N}}$ such that $b_t = c_{(d+1)t}$.

With these definitions we have:

Hilbert functions of subvarieties of $V_{n,d}$

Let $(h_t)_{t\in\mathbb{N}}$ be a sequence of non-negative integers such that $h_0 = 1$ and $h_1 = N + 1$. There exists a projective variety $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^N$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if $(h_t)_{t\in\mathbb{N}}$ is a differentiable $(d-1)$-sequence.

Theorem 1

Given $d, t, s \in \mathbb{N}$ such that $s \geq d^2t + \frac{d(d+3)}{2}$ we define the following two functions:

$$\mu_1(d,t,s) := d^2t + \frac{d(d+3)}{2} - s$$

$$\mu_2(d,t,s) := \begin{cases} \left\lfloor \frac{2d(t+1)+3-\sqrt{1+8\mu_1(d,t,s)}}{2} \right\rfloor, & \text{if } 1 \leq \mu_1(d,t,s) \leq \binom{d+1}{2} \\ dt - n, & \text{if } \binom{d+1}{2} + dn < \mu_1(d,t,s) \leq \binom{d+1}{2} + d(n+1), 0 \leq n \leq dt \end{cases}$$

Using the fact that reduced 0-dimensional varieties are always aCM and some properties of Hilbert functions of artinian ideals we proved the following theorem:

Hilbert functions of points on Veronese surfaces

Let $(h_t)_{t\in\mathbb{N}}$ be the Hilbert function of a finite set of m reduced points in $\mathbb{P}^{\frac{d(d+3)}{2}}$ and set

$$t_1 = \max\left\{t \mid h(t) = H_{V_{2,d}}(t)\right\} \qquad t_2 = \min\left\{t \mid h(t) = m\right\}.$$

Then there exists $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^N$, $|\mathbb{X}| = m$ such that $H_{\mathbb{X}}(t) = h_t$ if and only if the following conditions hold

- $$\mu_2(d, t_1, \Delta h_{t_1+1}) \geq \left\lceil \frac{\Delta h_{t_1+2}}{d} \right\rceil;$$
- For all $t_1 + 2 \leq t \leq t_2 - 1$
$$\left\lfloor \frac{\Delta h_t}{d} \right\rfloor \geq \left\lceil \frac{\Delta h_{t+1}}{d} \right\rceil.$$

An example

Let us consider the sequence $(h_t)_{t\in\mathbb{N}}$ defined as follows

t	0	1	2	3	4	5	6	7	8	9	10	11
h_t	1	36	120	253	435	666	946	1256	1531	1744	1956	2022

and $h_t = 2022$ for $t \geq 12$. Using a theorem of A.V. Geramita, P. Maroscia and L.G. Roberts, it is easy to check that this is the Hilbert function of a set of 2022 reduced points in $\mathbb{P}^{35}$. We ask whether there exists $\mathbb{X} \subseteq V_{2,7} \subseteq \mathbb{P}^{35}$ such that $H_{\mathbb{X}}(t) = h_t$ for all $t \geq 0$. To answer we use Theorem 1. First we determine t_1 and t_2. Since the Hilbert function of $V_{2,7}$ is $H_{V_{2,7}}(t) = \binom{2+7t}{2}$, we have that

t	0	1	2	3	4	5	6	7	8	9	10	11	12
$H_{V_{2,7}}$	1	36	120	253	435	666	946	1275	1653	2080	2556	3081	3655

so that $t_1 = 6$ and $t_2 = 11$. To determine $\mu_1(7, 6, \Delta h_{t_1+1})$ we compute Δh_{t_1+1}. We have that

t	0	1	2	3	4	5	6	7	8	9	10	11	12
Δh_t	1	35	84	133	182	231	280	310	275	213	212	66	0

and thus $\mu_1(7,6,310) = 7^2 \cdot 6 + \frac{7(7+3)}{2} - 310 = 19$. Finally, since $19 \leq \binom{7+1}{2} = 28$, we get

$$\mu_2(7,6,310) = \left\lfloor \frac{2 \cdot 7(6+1) + 3 - \sqrt{1 + 8 \cdot 19}}{2} \right\rfloor = 44.$$

To check conditions of Theorem 1 we compute $\lceil \frac{\Delta h_t}{7} \rceil$ and $\lfloor \frac{\Delta h_t}{7} \rfloor$ obtaining the following table

t	0	1	2	3	4	5	6	7	8	9	10	11	12
$\lceil \frac{\Delta h_t}{7} \rceil$	1	5	12	19	26	33	40	45	40	31	31	10	0
$\lfloor \frac{\Delta h_t}{7} \rfloor$	0	5	12	19	26	33	40	44	39	30	30	9	0

Since $\mu_2(7,6,310) = 44$ and $\lceil \frac{\Delta h_8}{7} \rceil = 40$ the first condition is satisfied. However the second condition is not satisfied for $t = 9$ and hence such an $\mathbb{X}$ does not exist.

Theorem 2

Using Theorem 1 we can characterize the complete intersections on Veronese surfaces.

Complete intersections on Veronese surfaces

If $\mathbb{X} \subseteq V_{2,d} \subseteq \mathbb{P}^N$ is a reduced complete intersection of type $(a_1, \ldots, a_r)$, with $a_1 \leq \cdots \leq a_r$ then one of the following holds:

- $(d, r, (a_1, a_2, \ldots, a_r)) = (2, 4, (1,1,1,2))$, that is $\mathbb{X}$ is a conic lying on $V_{2,2}$;
- $(d, r, (a_1, a_2, \ldots, a_r)) = (2, 5, (1,1,1,2,a_5))$, any $a_5 \in \mathbb{N}$, that is $\mathbb{X}$ is a set of $2a_5$ complete intersection points of a conic lying on $V_{2,2}$ and a hypersurface of degree a_5;
- $(d, r, (a_1, a_2, \ldots, a_r)) = (d, N, (1,1,\ldots,1))$ for any $d \geq 2$, that is $\mathbb{X}$ is a reduced point;
- $(d, r, (a_1, a_2, \ldots, a_r)) = (d, N, (1,1,\ldots,1,2)$ for any $d \geq 2$, that is $\mathbb{X}$ is a set of two reduced points.

Another example

If we want to find a conic $\mathcal{C} \subseteq V_{2,2}$ it suffices to consider $\nu_{2,2}(L)$ where $L \subseteq \mathbb{P}^2$ is a line. For example if we choose $L : x_2 = 0$ then we get

$$\mathcal{I}(\mathcal{C}) = (y_2, y_4, y_5, y_1^2 - y_0y_3)$$

that, indeed, is a complete intersection on $V_{2,2}$. Moreover, if we want to get a c.i. set of reduced points $\mathbb{X} \subseteq V_{2,2}$ with $|\mathbb{X}| = 2k$ we can take $\mathbb{X} = \mathcal{C} \cap \mathcal{V}(y_0^k - y_1^k)$.

What about the case $n \neq 2$?

We show that, except for the case $d = 2$, the only complete intersections lying on rational normal curves $V_{1,d}$ are the trivial ones, that is one single point or the set of two points. The case $V_{1,2}$, that is of a plane conic, is different. In fact, by cutting with any properly chosen curve, one will produce a complete intersection set of points. Inspired by these evidences we formulate the following conjecture:

Conjecture

Let $\mathbb{X} \subseteq V_{n,d} \subseteq \mathbb{P}^N$ be a reduced subvariety with $d > 1$. Then $\mathbb{X}$ is a complete intersection of type $(a_1, \ldots, a_r)$, with $a_1 \leq \cdots \leq a_r$ if and only if

- $r = N, a_1 = \ldots = a_N = 1$, any n, d, that is $\mathbb{X}$ is a reduced point;
- $r = N, a_1 = \ldots = a_{N-1} = 1, a_N = 2$, any n, d, that is $\mathbb{X}$ is a set of two reduced points;
- $r = N, a_1 = \ldots = a_{N-2} = 1, a_{N-1} = 2, a_N = b$, any n, $d = 2$, any $a \geq 2$, that is $\mathbb{X} = \mathcal{C} \cap H_b$ for $\mathcal{C} \subseteq V_{n,2}$ a conic and H_b a degree b hypersurface;
- $r = N - 1, a_1 = \ldots = a_{N-2} = 1, a_{N-1} = 2$, $d = 2$, any n, that is $\mathbb{X}$ is a conic.

We verify the conjecture in the case $n = 3, d = 2$ and prove the following, hopefully usefull, lemma:

Lemma

If Conjecture holds for all reduced zero dimensional subvariety of $V_{n,d}$, then it holds for all reduced subvarieties of $V_{n,d}$.