ON VARIETIES WITH ULRICH TWISTED CONORMAL BUNDLES

VINCENZO ANTONELLI, GIANFRANCO CASNATI, ANGELO FELICE LOPEZ AND DEBADITYA RAYCHAUDHURY

ABSTRACT. We study varieties $X \subset \mathbb{P}^r$ such that is $N_X^*(k)$ is an Ulrich vector bundle for some integer k. We first prove that such an X must be a curve. Then we give several examples of curves with $N_X^*(k)$ an Ulrich vector bundle.

1. INTRODUCTION

Let $X \subset \mathbb{P}^r$ be a smooth variety of dimension $n \geq 1$. Recall that a vector bundle $\mathcal{E}$ on X is called Ulrich if $H^i(\mathcal{E}(-p)) = 0$ for all $i \geq 0$ and $1 \leq p \leq n$. The importance of Ulrich vector bundles is well-known (see for example [ES, B, CMRPL] and references therein). While the main general problem about Ulrich vector bundles is their conjectural existence, another line of research around them is what are the consequences on the geometry of X in the presence of an Ulrich vector bundle. In this vein, we continue our study of which natural bundles, associated to X and to its embedding in $\mathbb{P}^r$, can be Ulrich up to some twist.

In previous papers, the third and fourth authors analyzed normal and tangent bundles, see [L, LR] (see also [BMPT]). In the present paper we study the following question: for which integers k one has that $N_X^*(k)$ is Ulrich?

A first simple consequence can be drawn: if X is degenerate, then $(X, H, k) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), 1)$, see Lemma 5.2.

On the other hand, suppose that X is nondegenerate. While in previous cases [L], [LR], examples of surfaces and threefolds appeared, we find a very different result for the conormal bundle. In fact we show that the answer to the above question is negative in dimension at least two.

Theorem 1.
Let $X \subset \mathbb{P}^r$ be a smooth nondegenerate variety such that $N_X^*(k)$ is Ulrich. Then X is a curve.

Now, for curves the situation is wide. First of all, there are many examples, at least in $\mathbb{P}^3$, stemming from some classical works [EL, EH, BE] (see Examples 8.1 and 8.2).

We first prove that there is a sharp bound for the degree of a curve having Ulrich twisted conormal bundle.

Theorem 2.
Let $C \subset \mathbb{P}^{c+1}$ be a smooth nondegenerate curve of degree d and codimension $c \geq 1$ such that $N_C^*(k)$ is Ulrich. Then $c \geq 2$ and

$$d \geq \frac{c+2}{2k+c}\binom{k+c}{c+1}. \tag{1.1}$$

Moreover this bound is sharp for $c = 2$ and $k \equiv 1, 3 \pmod 6$.

On the other hand, the examples mentioned above, Examples 8.1 and 8.2, are all subcanonical curves in $\mathbb{P}^3$. We show that neither the fact of being subcanonical, nor of lying in $\mathbb{P}^3$, is a necessary condition, by producing examples, for unbounded genus, of non-subcanonical curves in $\mathbb{P}^3$ and in $\mathbb{P}^4$.

Theorem 3.
(i) Let $X \subset \mathbb{P}^3$ be a general nonspecial curve of genus g and degree $d = 2g - 2$. Then $N_X^*(4)$ is Ulrich and X is not subcanonical.

The first three authors are members of the GNSAGA group of INdAM. The third author was partially supported by PRIN "Advances in Moduli Theory and Birational Classification".

Mathematics Subject Classification : Primary 14F06. Secondary 14H60, 14H10, 14J60.

(ii) Let $X \subset \mathbb{P}^4$ be a general curve of genus $g \geq 29$ and degree $d = 5g - 5$. Then $N_X^*(3)$ is Ulrich and X is not subcanonical.

Note that the nonspecial curve of genus 5 and degree 8 in (i) of the above theorem, is a non-subcanonical curve realizing equality in Theorem 2.

We believe that many more examples of curves with $N_X^*(k)$ Ulrich, can probably be constructed, by refining the methods, such as using elementary modifications and special degenerations as in [ALY, BR, R]. One possible direction is given in Remark 8.7.

2. Notation

Throughout the paper we work over the field $\mathbb{C}$ of complex numbers. A *variety* is by definition an integral separated scheme of finite type over $\mathbb{C}$. A *curve* (respectively a *surface*) is a variety of dimension 1 (resp. 2). Moreover, we henceforth establish the following:

Notation 2.1.

- $X \subset \mathbb{P}^r$ is a smooth closed variety of dimension $n \geq 1$ and codimension $c = r - n \geq 1$.
- H is a hyperplane divisor.
- $N_X := N_{X/\mathbb{P}^r}$ is the normal bundle.
- For any sheaf $\mathcal{G}$ on X we set $\mathcal{G}(l) = \mathcal{G}(lH)$.
- $d = H^n$ is the degree of X.
- C is a general curve section of X under H.
- S is a general surface section of X under H, when $n \geq 2$.
- $g = g(C) = \frac{1}{2}[K_X H^{n-1} + (n-1)d] + 1$ is the sectional genus of X.
- For $1 \leq i \leq n-1$, let $H_i \in |H|$ be general divisors and set $X_n := X$ and $X_i = H_1 \cap \cdots \cap H_{n-i}$. In particular $X_1 = C, X_2 = S$.
- $s(X) = \min\{s \geq 1 : H^0(\mathcal{J}_{X/\mathbb{P}^r}(s)) \neq 0\}$.

We will also let $V = H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ and consider the exact sequences

$$0 \to \Omega^1_{\mathbb{P}^r|X} \to V \otimes \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0 \tag{2.1}$$

and

$$0 \to N_X^* \to \Omega^1_{\mathbb{P}^r|X} \to \Omega^1_X \to 0. \tag{2.2}$$

3. A general fact about projective varieties

We record here a simple but useful fact.

Lemma 3.1. *Let $X \subset \mathbb{P}^r$ be a smooth variety of dimension $n \geq 1$. If $H^0(N_X^*(l)) = 0$ and $\pi : X \to \overline{X} \subset \mathbb{P}^m$ is an isomorphic projection, then $l \leq \min\{s(X) - 1, s(\overline{X}) - 1\}$.*

Proof. Set $s = s(X)$ and suppose that $l \geq s$, so that $H^0(N_X^*(s)) = 0$. Now the exact sequence

$$0 \to \mathcal{J}^2_{X/\mathbb{P}^r}(s) \to \mathcal{J}_{X/\mathbb{P}^r}(s) \to N_X^*(s) \to 0$$

implies that $h^0(\mathcal{J}^2_{X/\mathbb{P}^r}(s)) = h^0(\mathcal{J}_{X/\mathbb{P}^r}(s)) > 0$, hence there is a hypersurface F of degree s such that $X \subseteq \mathrm{Sing}(F)$. But then a partial derivative of the equation of F would give a hypersurface of degree $s-1$ containing X, a contradiction. Therefore

$$l \leq s(X) - 1. \tag{3.1}$$

Now let $\pi : X \to \overline{X} \subset \mathbb{P}^m$ be an isomorphic projection. We have an exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(1)^{\oplus(r-m)} & \xrightarrow{\cong} & \mathcal{O}_X(1)^{\oplus(r-m)} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X(1)^{\oplus(r+1)} & \longrightarrow & T_{\mathbb{P}^r|X} & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*\mathcal{O}_{\overline{X}} & \longrightarrow & \pi^*\mathcal{O}_{\overline{X}}(1)^{\oplus(m+1)} & \longrightarrow & \pi^*T_{\mathbb{P}^m|\overline{X}} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and therefore also an exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_X(1)^{\oplus(r-m)} & \xrightarrow{\cong} & \mathcal{O}_X(1)^{\oplus(r-m)} & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_X & \longrightarrow & T_{\mathbb{P}^r|X} & \longrightarrow & N_X & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi^*T_{\overline{X}} & \longrightarrow & \pi^*T_{\mathbb{P}^m|\overline{X}} & \longrightarrow & \pi^*N_{\overline{X}/\mathbb{P}^m} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} .$$

Hence, we deduce an exact sequence

$$0 \to \pi^* N^*_{\overline{X}/\mathbb{P}^m} \to N^*_X \to \mathcal{O}_X(-1)^{\oplus(r-m)} \to 0. \tag{3.2}$$

Since $H^0(N^*_X(l)) = 0$ we get that $H^0(\overline{X}, N^*_{\overline{X}/\mathbb{P}^m}(l)) = H^0(X, \pi^* N^*_{\overline{X}/\mathbb{P}^m}(l)) = 0$. Hence applying (3.1) to $\overline{X} \subset \mathbb{P}^m$ we get that $l \le s(\overline{X}) - 1$ and the lemma is proved. □

4. Generalities on Ulrich bundles

We collect here some well-known facts about Ulrich bundles, to be used sometimes later.

Definition 4.1. Let $\mathcal{E}$ be a vector bundle on X. We say that $\mathcal{E}$ is *Ulrich* for (X, H) if $H^i(\mathcal{E}(-p)) = 0$ for all $i \ge 0$ and $1 \le p \le n$.

We have

Lemma 4.2. *Let $\mathcal{E}$ be a rank t Ulrich vector bundle for (X, H). Then*

(i) $c_1(\mathcal{E})H^{n-1} = \frac{t}{2}[K_X + (n+1)H]H^{n-1}$.
(ii) *$\mathcal{E}^*(K_X + (n+1)H)$ is also Ulrich for (X, H).*
(iii) *$\mathcal{E}$ is globally generated.*
(iv) *$\mathcal{E}$ is arithmetically Cohen-Macaulay (aCM), that is $H^i(\mathcal{E}(j)) = 0$ for $0 < i < n$ and all $j \in \mathbb{Z}$.*
(v) *$\mathcal{E}_{|Y}$ is Ulrich on a smooth hyperplane section Y of X.*
(vi) *$\mathcal{O}_X(l)$ is Ulrich if and only if $(X, H, l) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), 0)$.*

Proof. Well-known. For (i)-(v) see for example [LR, Lemma 3.2]. As for (vi), it is obvious that $\mathcal{O}_{\mathbb{P}^n}$ is Ulrich for $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. Vice versa, if $\mathcal{O}_X(l)$ is Ulrich, then it is globally generated by (iii), so that $l \geq 0$. But also $H^0(\mathcal{O}_X(l-1)) = 0$, hence $l = 0$. It follows by [LR, Lemma 3.2(vii)] that $d = h^0(\mathcal{O}_X) = 1$, so that $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$. □

Lemma 4.3. *Let $X \subset \mathbb{P}^r$ be a smooth variety of dimension $n \geq 3$. Let $\mathcal{E}$ be a vector bundle on X and let Y be a smooth hyperplane section of X. If $\mathcal{E}_{|Y}$ is Ulrich, then $\mathcal{E}$ is Ulrich.*

Proof. For $j \in \mathbb{Z}$ consider the exact sequence

$$0 \to \mathcal{E}(j-1) \to \mathcal{E}(j) \to \mathcal{E}_{|Y}(j) \to 0. \tag{4.1}$$

If $2 \leq i \leq n-2$ we have that $H^{i-1}(\mathcal{E}_{|Y}(j)) = H^i(\mathcal{E}_{|Y}(j)) = 0$ for any $j \in \mathbb{Z}$ by Lemma 4.2(iv). Hence (4.1) gives that $h^i(\mathcal{E}(j-1)) = h^i(\mathcal{E}(j))$ for any $j \in \mathbb{Z}$. On the other hand $h^i(\mathcal{E}(j)) = 0$ for $j \gg 0$ and it follows that $h^i(\mathcal{E}(j)) = 0$ for any $j \in \mathbb{Z}$ and $2 \leq i \leq n-2$.

Suppose now that $i \in \{0, 1\}$ and $j \leq -1$. We have that $H^0(\mathcal{E}_{|Y}(j)) = 0$ and, since $n-1 \geq 2$, also that $H^1(\mathcal{E}_{|Y}(j)) = 0$ by Lemma 4.2(iv). Hence (4.1) gives that $h^i(\mathcal{E}(j-1)) = h^i(\mathcal{E}(j))$. On the other hand, by Serre duality, $h^i(\mathcal{E}(j)) = h^{n-i}(\mathcal{E}^*(K_X - jH)) = 0$ for $j \ll 0$ and therefore

$$h^i(\mathcal{E}(j)) = 0 \text{ for } i \in \{0, 1\} \text{ and } j \leq -1. \tag{4.2}$$

Now let $\mathcal{E}' = \mathcal{E}^*(K_X + (n+1)H)$. Then $\mathcal{E}'_{|Y} = \mathcal{E}^*_{|Y}(K_Y + nH_{|Y})$ is also Ulrich by Lemma 4.2(ii). Therefore (4.2) implies that $h^i(\mathcal{E}'(j)) = 0$ for $i \in \{0, 1\}$ and $j \leq -1$. By Serre duality we get that $h^{n-i}(\mathcal{E}(-n-1-j)) = h^i(\mathcal{E}^*(K_X + (n+1+j)H)) = h^i(\mathcal{E}'(j)) = 0$ for $i \in \{0, 1\}$ and $j \leq -1$. But this is the same as $h^s(\mathcal{E}(l)) = 0$ for $s \in \{n-1, n\}$ and $l \geq -n$.

Thus we have proved that $H^i(\mathcal{E}(-p)) = 0$ for $i \geq 0$ and $1 \leq p \leq n$, that is $\mathcal{E}$ is Ulrich. □

5. Ulrich conormal bundles

In this section we will draw some very useful consequences and facts for varieties $X \subset \mathbb{P}^r$ such that $N_X^*(k)$ is Ulrich.

The first one is a reduction via hyperplane sections.

Lemma 5.1. *Let $X \subset \mathbb{P}^{c+n}$ be a smooth variety of dimension n and codimension $c \geq 1$. If $n \geq 2$ and $N_X^*(k)$ is Ulrich, then $N^*_{X_i/\mathbb{P}^{c+i}}(k)$ is Ulrich for all $i \in \{1, \dots, n-1\}$. Vice versa, if $n \geq 3$ and $N^*_{X_i/\mathbb{P}^{c+i}}(k)$ is Ulrich for some $i \in \{2, \dots, n-1\}$, then $N^*_{X_j/\mathbb{P}^{c+j}}(k)$ is Ulrich for all $j \in \{2, \dots, n\}$ (hence in particular so is $N_X^*(k)$).*

Proof. Recall that if $Y \subset \mathbb{P}^m$ is smooth and Z is a smooth hyperplane section, then

$$(N_{Y/\mathbb{P}^m})_{|Z} \cong N_{Z/\mathbb{P}^{m-1}}.$$

Now if $N_X^*(k)$ is Ulrich, then so are all $N^*_{X_i/\mathbb{P}^{c+i}}(k)$ by Lemma 4.2(v).

Vice versa, if $N^*_{X_i/\mathbb{P}^{c+i}}(k)$ is Ulrich for some $i \in \{2, \dots, n-1\}$, then $N^*_{X_{i+1}/\mathbb{P}^{c+i+1}}(k)$ is Ulrich by Lemma 4.3. Repeating the argument we get that $N^*_{X_j/\mathbb{P}^{c+j}}(k)$ is Ulrich for all $j \in \{2, \dots, n\}$. □

We now deal with X degenerate in $\mathbb{P}^r$.

Lemma 5.2. *Let $X \subset \mathbb{P}^r$ be a smooth degenerate variety of dimension $n \geq 1$. Then $N_X^*(k)$ is Ulrich if and only if $(X, H, k) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), 1)$.*

Proof. If $(X, H) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ then $N_X^*(1) = \mathcal{O}_{\mathbb{P}^n}^{\oplus c}$ is Ulrich.

Vice versa assume that $N_X^*(k)$ is Ulrich. Since X is degenerate, $N_X^*(k)$ has $\mathcal{O}_X(k-1)$ as a direct summand. Therefore also $\mathcal{O}_X(k-1)$ is Ulrich and Lemma 4.2(vi) gives that $(X, H, k) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), 1)$. □

In the sequel we will then consider only nondegenerate varieties.

We start by collecting some cohomological and numerical conditions.

Lemma 5.3. (cohomological conditions)

Let $X \subset \mathbb{P}^{c+n}$ be a smooth nondegenerate variety of dimension $n \geq 1$ and codimension $c \geq 1$. If $N_X^*(k)$ is Ulrich, we have:

(i) *$H^n(\mathcal{O}_X(l)) = 0$ for every $l \ge k-n-1$.*
(ii) *If $n \ge 2$, then $X \subset \mathbb{P}^{c+n}$ is projectively normal.*
(iii) *If $n \ge 2$ then $q(X) = 0$.*
(iv) *$k \le \min\{s(X), s(\overline{X})\}$, where $\overline{X} \subset \mathbb{P}^r$ is any isomorphic projection of X.*

Proof. By hypothesis $N_X^*(k)$ is Ulrich, hence it is aCM by Lemma 4.2(iv).

To see (i) we just need to prove that $H^n(\mathcal{O}_X(k-n-1)) = 0$. Assume that $H^n(\mathcal{O}_X(k-n-1)) \ne 0$, that is, by Serre duality, $H^0(K_X + (n+1-k)H) \ne 0$. Then we have an inclusion

$$N_X(-1) \hookrightarrow N_X(K_X + (n-k)H)$$

and, since $N_X(-1)$ is globally generated, we get that $h^0(N_X(K_X + (n-k)H)) \ne 0$. On the other hand, $N_X(K_X + (n+1-k)H)$ is Ulrich by Lemma 4.2(ii), hence $h^0(N_X(K_X + (n-k)H)) = 0$. This contradiction proves (i). To see (ii), let $P^1(\mathcal{O}_X(1))$ be the sheaf of principal parts and consider, as in [E, Proof of Thm. 2.4], the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N_X^*(1) & \xrightarrow{\cong} & N_X^*(1) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \Omega^1_{\mathbb{P}^{c+n}}(1)_{|X} & \to & V \otimes \mathcal{O}_X & \to & \mathcal{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \cong & & \\
0 & \to & \Omega^1_X(1) & \to & P^1(\mathcal{O}_X(1)) & \to & \mathcal{O}_X(1) & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} .$$

Pick an integer $l \ge 0$. Tensoring the above diagram by $\mathcal{O}_X(l)$ and observing that

$$P^1(\mathcal{O}_X(1)) \otimes \mathcal{O}_X(l) \cong P^1(\mathcal{O}_X(l+1))$$

by [E, (2.2)], we get the commutative diagram

$$\begin{array}{ccc}
V \otimes H^0(\mathcal{O}_X(l)) & & \\
\downarrow f_l & \searrow^{h_l} & \\
H^0(P^1(\mathcal{O}_X(l+1))) & \xrightarrow{g_l} & H^0(\mathcal{O}_X(l+1)) \\
\downarrow & & \\
H^1(N_X^*(l+1)). & &
\end{array} \tag{5.1}$$

Now we have that $H^1(N_X^*(l+1)) = 0$ since N_X^* is aCM and $n \ge 2$. Hence f_l is surjective for every $l \ge 0$ and so is g_l by [E, Prop. 2.3]. It follows by (5.1) that h_l is surjective for every $l \ge 0$. Moreover the commutative diagram

$$\begin{array}{ccc}
V \otimes H^0(\mathcal{O}_{\mathbb{P}^{c+n}}(l)) & \longrightarrow & H^0(\mathcal{O}_{\mathbb{P}^{c+n}}(l+1)) \\
\downarrow \mathrm{Id}_V \otimes r_l & & \downarrow r_{l+1} \\
V \otimes H^0(\mathcal{O}_X(l)) & \xrightarrow{h_l} & H^0(\mathcal{O}_X(l+1))
\end{array} \tag{5.2}$$

shows by induction that $r_l : H^0(\mathcal{O}_{\mathbb{P}^{c+n}}(l)) \to H^0(\mathcal{O}_X(l))$ is surjective for every $l \ge 0$, so that $X \subset \mathbb{P}^r$ is projectively normal, that is (ii).

To see (iii) observe that, (2.1) gives an exact sequence

$$0 \longrightarrow H^0(\Omega^1_{\mathbb{P}^{c+n}}(1)_{|X}) \longrightarrow V \xrightarrow{f} H^0(\mathcal{O}_X(1)) \tag{5.3}$$

and f is injective since X is nondegenerate, hence $H^0(\Omega^1_{\mathbb{P}^{c+n}}(1)_{|X}) = 0$. Now $N_X^*(k)$ is aCM and therefore $H^1(N_X^*(1)) = 0$. Then the exact sequence

$$0 \to N_X^*(1) \to \Omega^1_{\mathbb{P}^{c+n}}(1)_{|X} \to \Omega^1_X(1) \to 0 \tag{5.4}$$

shows that $H^0(\Omega^1_X(1)) = 0$, hence, in particular $q(X) = h^0(\Omega^1_X) = 0$. This proves (iii). Finally (iv) follows by Lemma 3.1 since, $N_X^*(k)$ being Ulrich, we have that $H^0(N_X^*(k-1)) = 0$. □

Lemma 5.4. (numerical conditions)
Let $X \subset \mathbb{P}^{c+n}$ be a smooth nondegenerate variety of dimension $n \geq 1$ and codimension $c \geq 1$. If $N_X^*(k)$ is Ulrich, we have:

(i) $[(k-2)c-2]d = (c+2)(g-1)$.
(ii) $c \geq 2$.
(iii) $k \geq 3$.

Proof. By hypothesis $N_X^*(k)$ is Ulrich. By (2.1) and (2.2) we see that $c_1(N_X^*(k)) = -K_X - (c+n+1-kc)H$. Hence Lemma 4.2(i) implies

$$-(K_X + (c+n+1-kc)H)H^{n-1} = \frac{c}{2}\left(K_XH^{n-1} + (n+1)d\right)$$

and this gives $K_XH^{n-1} = (2k-n-3-\frac{4(k-1)}{c+2})d$. But also $K_XH^{n-1} = 2(g-1)-(n-1)d$ and we get (i). As for (ii), if $c=1$ then $N_X^*(k) = \mathcal{O}_X(k-d)$ and Lemma 4.2(vi) gives that $X \subset \mathbb{P}^r$ is a linear space, a contradiction. This proves (ii). To see (iii), since X is nondegenerate, we get from (2.1) that $H^0(\Omega^1_{\mathbb{P}^{c+n}|X}(1)) = 0$. Now (2.2) gives that $h^0(\Omega^1_{\mathbb{P}^{c+n}|X}(k)) \geq h^0(N_X^*(k)) > 0$ by Lemma 4.2(iii). Hence $k \geq 2$. But if $k=2$ then (i) gives that $g=0$ and $d = \frac{c+2}{2}$. As it is well known, $d \geq c+1$, giving a contradiction. Thus (iii) holds. □

6. Properties of the surface section

We deduce here some very useful properties of the surface section of some $X \subset \mathbb{P}^r$ such that $N_X^*(k)$ is Ulrich.

Lemma 6.1. *Let $X \subset \mathbb{P}^{c+n}$ be a smooth nondegenerate variety of dimension $n \geq 2$ and codimension $c \geq 1$. If $N_X^*(k)$ is Ulrich, the following inequality holds for the surface section S:*

$$\chi(\mathcal{O}_S) \geq \frac{d}{2(c+2)(c+3)(3c+4)(c+12)}[(3c^4+43c^3+86c^2+24c)k^2-(15c^4+229c^3+624c^2+432c)k+ \\ +18c^4+296c^3+1070c^2+1368c+576].$$

Proof. Note that $N^*_{S/\mathbb{P}^{c+2}}(k)$ is Ulrich by Lemma 5.1. Therefore, Lemma 5.3(iii) implies that

$$q(S) = 0. \tag{6.1}$$

Next, computing the Chern classes of $N^*_{S/\mathbb{P}^{c+2}}(k)$ and applying [C, (2.2)], we get

$$cK_S^2 + (c+12)c_2(S) = \frac{6d}{c+2}[c(c+2)k^2 - c(5c+12)k + 6c^2 + 20c + 12]. \tag{6.2}$$

Now note that $N^*_{S/\mathbb{P}^{c+2}}(k)$ is semistable by [CH, Thm. 2.9], hence so is $N^*_{S/\mathbb{P}^{c+2}}$ and the Bogomolov inequality is

$$d(c+3)(4k-7-\frac{8(k-1)}{c+2}) + (c+1)K_S^2 \geq 2cc_2(S). \tag{6.3}$$

Then (6.2) and (6.3) give

$$K_S^2 \geq \frac{d}{(c+2)(c+3)(3c+4)}[(12c^3+24c^2)k^2-(64c^3+204c^2+144c)k+79c^3+351c^2+486c+216]. \tag{6.4}$$

Finally the inequality on $\chi(\mathcal{O}_S)$ in the statement follows by (6.2), (6.4) and Noether's formula. □

7. Proofs of main theorems

We start by proving Theorem 2.

Proof of Theorem 2. Since $N_C^*(k)$ is Ulrich, it follows that $c \geq 2$ by Lemma 5.4(ii) and $k \leq s(C)$ by Lemma 5.3(iv). Also Lemma 5.3(i) gives $H^1(\mathcal{O}_C(k-2)) = 0$. Therefore we have that $H^0(\mathcal{J}_{C/\mathbb{P}^{c+1}}(k-1)) = H^1(\mathcal{O}_C(k-1)) = 0$ and the exact sequence

$$0 \to \mathcal{J}_{C/\mathbb{P}^{c+1}}(k-1) \to \mathcal{O}_{\mathbb{P}^{c+1}}(k-1) \to \mathcal{O}_C(k-1) \to 0$$

together with Riemann-Roch, shows that

$$\binom{k+c}{c+1} = h^0(\mathcal{O}_{\mathbb{P}^{c+1}}(k-1)) \leq h^0(\mathcal{O}_C(k-1)) = d(k-1) - g + 1. \tag{7.1}$$

Also, $g - 1 = \frac{(k-2)c-2}{c+2} d$ by Lemma 5.4(i) and replacing in (7.1) we get (1.1).

Finally, sharpness for $c = 2$ and $k \equiv 1, 3 \pmod 6$ follows by [BE, Examples, p. 88], see Example 8.2. □

Next, we prove Theorem 1.

Proof of Theorem 1. Suppose that $n \geq 2$. In order to simplify the calculations we set

$$A = (3c^4+43c^3+86c^2+24c)k^2 - (15c^4+229c^3+624c^2+432c)k + 18c^4+296c^3+1070c^2+1368c+576$$

so that it follows by Lemma 6.1 that

$$\chi(\mathcal{O}_S) \geq \frac{dA}{2(c+2)(c+3)(3c+4)(c+12)}. \tag{7.2}$$

Now Lemma 5.1 implies that $N^*_{S/\mathbb{P}^{c+2}}(k)$ is Ulrich. Hence $k \leq s(S)$ by Lemma 5.3(iv), $H^2(\mathcal{O}_S(k-2)) = 0$ by Lemma 5.3(i) and $S \subset \mathbb{P}^{c+2}$ is projectively normal by Lemma 5.3(ii). Therefore, we have that

$$H^0(\mathcal{J}_{S/\mathbb{P}^{c+2}}(k-2)) = H^1(\mathcal{J}_{S/\mathbb{P}^{c+2}}(k-2)) = H^2(\mathcal{O}_S(k-2)) = 0$$

and the exact sequence

$$0 \to \mathcal{J}_{S/\mathbb{P}^{c+2}}(k-2) \to \mathcal{O}_{\mathbb{P}^{c+2}}(k-2) \to \mathcal{O}_S(k-2) \to 0$$

together with Riemann-Roch and Lemma 5.4(i), shows that

$$\begin{aligned}\binom{k+c}{c+2} &= h^0(\mathcal{O}_{\mathbb{P}^{c+2}}(k-2)) = h^0(\mathcal{O}_S(k-2)) = \chi(\mathcal{O}_S(k-2)) + h^1(\mathcal{O}_S(k-2)) \geq \\ &\geq \chi(\mathcal{O}_S) + \frac{d(k-2)}{2}\left(3 - k + \frac{4(k-1)}{c+2}\right).\end{aligned} \tag{7.3}$$

Setting

$$B = A + (k-2)(c+3)(3c+4)(c+12)[(3-k)(c+2)+4k-4] = 2(c+12)(k-1)(8ck+12k+5c^2+7c)$$

we see, using (7.2), that (7.3) implies

$$\binom{k+c}{c+2} \geq \frac{dB}{2(c+2)(c+3)(3c+4)(c+12)} = \frac{d(k-1)(8ck+12k+5c^2+7c)}{(c+2)(c+3)(3c+4)}. \tag{7.4}$$

Since $k \geq 3$ by Lemma 5.4(iii), we see that (7.4) gives

$$d \leq \frac{(c+2)(c+3)(3c+4)}{(k-1)(8ck+12k+5c^2+7c)}\binom{k+c}{c+2}.$$

Since $N_X^*(k)$ is Ulrich, it follows by Lemma 5.1 that $N^*_{C/\mathbb{P}^{c+1}}(k)$ is Ulrich, hence, using Theorem 2, we find that

$$\frac{c+2}{2k+c}\binom{k+c}{c+1} \leq d \leq \frac{(c+2)(c+3)(3c+4)}{(k-1)(8ck+12k+5c^2+7c)}\binom{k+c}{c+2}$$

that is equivalent to $2c(c+1)(c+k+1) \leq 0$, a contradiction. □

8. Curves

In this section we construct some examples of curves $C \subset \mathbb{P}^{c+1}$ such that $N_C^*(k)$ is Ulrich.

First, we give a reinterpretation of two known cases.

Example 8.1. For every integer $d \geq 5$ there exists a smooth elliptic curve $C \subset \mathbb{P}^3$ such that $N_C^*(3)$ is Ulrich.

In fact, it follows by [EL, Prop. §8, page 278] and [EH, Thm. 2(b)] that for every integer $d \geq 5$ there exists a smooth elliptic curve $C \subset \mathbb{P}^3$ such that $H^0(N_C(-2)) = 0$. Since $\chi(N_C(-2)) = 0$ we get that $N_C(-2)$ is Ulrich and then also $N_C^*(3)$ is Ulrich by Lemma 4.2(ii).

Example 8.2. There are many subcanonical curves $X \subset \mathbb{P}^3$ with $H^0(N_X(-2)) = 0$ (see for example [BE, Examples, p. 88]), hence with $N_X(-1)$ Ulrich. Therefore, since $K_X = eH$ for some $e \in \mathbb{Z}$, we also have by Lemma 4.2(ii), that $N_X^*(e+3)$ is Ulrich.

We now describe the curves in [BE, Examples, p. 88]. Let $h \in \mathbb{Z}$ such that $h \geq 1$, let $c_2 = h(3h+2)$ and let $t = 3h+1$ (resp. $h \geq 0, c_2 = 3h^2+4h+1$ and $t = 3h+2$). Then in loc. cit, there are examples of smooth curves $X \subset \mathbb{P}^3$ with $N_X^*(e+3)$ Ulrich, $d = t^2 + c_2$ and $g = (t-2)d+1$. We have $e = \frac{2g-2}{d} = 2t-4$, hence $N_X^*(k)$ is Ulrich with $k = e+3 = 2t-1$. In the first case, $t = 3h+1$, we have $h = \frac{k-1}{6}, k \equiv 1 (\text{mod } 6)$ and $d = \frac{1}{3}(k^2+2k)$. In the second case, $t = 3h+2$, we have $h = \frac{k-3}{6}, k \equiv 3 (\text{mod } 6)$ and again $d = \frac{1}{3}(k^2+2k)$. In particular they have unbounded k.

The examples above are all subcanonical curves. In order to construct non-subcanonical ones, we will proceed below by degeneration. We will apply a simple version, in higher rank, of some results in [HH].

Definition 8.3. Let X be a smooth curve, let $\mathcal{N}$ be a rank t vector bundle on X and let $\mathcal{K} \subset \mathbb{P}(\mathcal{N})$ be a finite subset such that $\pi_{|\mathcal{K}} : \mathcal{K} \to \pi(\mathcal{K})$ is injective, where $\pi : \mathbb{P}(\mathcal{N}) \to X$. Then we define

$$\operatorname{elm}_{\mathcal{K}}^- \mathcal{N} = \pi_*(\mathcal{J}_{\mathcal{K}/\mathbb{P}(\mathcal{N})} \otimes \mathcal{O}_{\mathbb{P}(\mathcal{N})}(1)).$$

We have

Lemma 8.4. *Let X be a smooth curve, let $\mathcal{M}, \mathcal{N}$ be rank t vector bundles and $\mathcal{L}$ a line bundle on X. Let $\mathcal{K} \subset \mathbb{P}(\mathcal{N})$ be a finite subset such that $\pi_{|\mathcal{K}} : \mathcal{K} \to \pi(\mathcal{K})$ is injective, where $\pi : \mathbb{P}(\mathcal{N}) \to X$. Then:*

(i) $(\operatorname{elm}_{\mathcal{K}}^- \mathcal{N}) \otimes \mathcal{L} \cong \operatorname{elm}_{\mathcal{K}}^- (\mathcal{N} \otimes \mathcal{L})$.
(ii) *If $\mathcal{K}$ is general in $\mathbb{P}(\mathcal{N})$, then $h^0(\operatorname{elm}_{\mathcal{K}}^- \mathcal{N}) = \max\{0, h^0(\mathcal{N}) - \sharp \mathcal{K}\}$.*
(iii) *If $0 \to \mathcal{M} \to \mathcal{N} \to \mathcal{O}_{\pi(\mathcal{K})} \to 0$ is an exact sequence, then $\mathcal{N}^* \cong \operatorname{elm}_{\mathcal{K}}^- (\mathcal{M}^*)$.*

Proof. (i) and (ii) follow exactly as in [HH]. Also, as in [HH], we have an exact sequence

$$0 \to \operatorname{elm}_{\mathcal{K}}^- \mathcal{N} \to \mathcal{N} \to \mathcal{O}_{\pi(\mathcal{K})} \to 0. \tag{8.1}$$

As for (iii), observe that dualizing the exact sequence

$$0 \to \mathcal{O}_X(-\pi(\mathcal{K})) \to \mathcal{O}_X \to \mathcal{O}_{\pi(\mathcal{K})} \to 0$$

we get that $\mathcal{E}xt^1(\mathcal{O}_{\pi(\mathcal{K})}, \mathcal{O}_X) \cong \mathcal{O}_{\pi(\mathcal{K})}$, hence dualizing the exact sequence in (iii), we get the exact sequence

$$0 \to \mathcal{N}^* \to \mathcal{M}^* \to \mathcal{O}_{\pi(\mathcal{K})} \to 0$$

that defines an embedding of $\mathcal{K}$ in $\mathbb{P}(\mathcal{M}^*)$ and therefore $\mathcal{N}^* \cong \operatorname{elm}_{\mathcal{K}}^- (\mathcal{M}^*)$ by (8.1). □

We can now state a smoothing lemma that allows to construct curves with $N_X^*(k)$ Ulrich by degeneration.

Lemma 8.5. *Let $c \geq 2, s \geq 1$, let $P_1, \dots, P_s \in \mathbb{P}^{c+1}$ be general points and set $D = \{P_1, \dots, P_s\}$. For each $i \in \{1, \dots, s\}$, let L_i, L_i' be general lines in $\mathbb{P}^{c+1}$, each passing through P_i. Suppose that there are two smooth curves $Y, Z \subset \mathbb{P}^{c+1}$ of degree d_Y, d_Z and genus g_Y, g_Z such that:*

(i) *$D = Y \cap Z$ is a general effective divisor of degree s on both.*
(ii) *$T_{P_i} Y = L_i, T_{P_i} Z = L_i', 1 \leq i \leq s$.*
(iii) *Either*
  (iii-1) *$N_{Z/\mathbb{P}^{c+1}}$ is stable, or*

(iii-2) $H^0(N_Z(K_Z-(k-1)H))=0$.

(iv) *Either*

(iv-1) $N_{Y/\mathbb{P}^{c+1}}$ *is semistable if* $s>c(g_Y-1)$ *and is stable if* $s=c(g_Y-1)$*, or*

(iv-2) $H^0(N_Y(K_Y+D-(k-1)H))=0$.

(v) $(c+2)(g_Z-1)-[c(k-2)-2]d_Z+s=0$.

(vi) $(c+2)(g_Y-1)-[c(k-2)-2]d_Y+(c+1)s=0$.

(vii) $s\ge\min\{c(g_Z-1),c(g_Y-1)\}$.

Let $X'=Y\cup Z$ *and assume that it is smoothable in* $\mathbb{P}^{c+1}$*. Then a general smoothing* $X\subset\mathbb{P}^{c+1}$ *of* X' *is such that* $\deg X=d_Y+d_Z, g(X)=g_Y+g_Z+s-1$ *and* $N^*_{X/\mathbb{P}^{c+1}}(k)$ *is Ulrich.*

Proof. It follows by (v) and (vi) that

$$\chi(N_Z(K_Z-(k-1)H))=\chi(N_Y(K_Y+D-(k-1)H))=-s. \tag{8.2}$$

Also (vii) is equivalent to

$$\mu(N_Z(K_Z-(k-1)H))\le 0, \mu(N_Y(K_Y+D-(k-1)H))\le 0.$$

Now, it follows by (iii), in either case, that

$$H^0(N_Z(K_Z-(k-1)H))=0. \tag{8.3}$$

Next, we prove that under hypothesis (iv) we have that $H^0(N_Y(K_Y+D-(k-1)H))=0$. In fact, assume (iv-1). Then, either $s>c(g_Y-1)$, hence $\mu(N_Y(K_Y+D-(k-1)H))<0$ and N_Y is semistable, or, by (vii), $s=c(g_Y-1)$ and N_Y is stable. In both cases it follows that

$$H^0(N_Y(K_Y+D-(k-1)H))=0. \tag{8.4}$$

Hence (8.3), (8.4) and (8.2) imply that

$$h^1(N_Z(K_Z-(k-1)H))=-\chi(N_Z(K_Z-(k-1)H))=s \tag{8.5}$$

and

$$h^1(N_Y(K_Y+D-(k-1)H))=-\chi(N_Y(K_Y+D-(k-1)H))=s. \tag{8.6}$$

In order to prove that $N^*_X(k)$ is Ulrich, observe that (v) and (vi) imply that $\chi(N^*_X(k-1))=0$. Therefore it will suffice to prove that $H^0(N^*_X(k-1))=0$, or, by duality, that $H^1(N_X\otimes\omega_X(-k+1))=0$. This in turn will follow, by semicontinuity, if we prove that

$$H^1(N_{X'}\otimes\omega_{X'}(-k+1))=0. \tag{8.7}$$

Since $\omega_{X'|Y}\cong\omega_Y(D)$ and $\omega_{X'|Z}\cong\omega_Z(D)$, to show (8.7), we will use the exact sequence

(8.8)

$$0\to N_{X'}\otimes\omega_{X'}(-k+1)\to N_{X'|_Y}(K_Y+D-(k-1)H)\oplus N_{X'|_Z}(K_Z+D-(k-1)H)\to N_{X'|_D}\otimes\omega_{X'}(-k+1)\to 0$$

and prove that

$$H^1(N_{X'|_Y}(K_Y+D-(k-1)H))=0 \tag{8.9}$$

and

$$H^1(N_{X'|_Z}(K_Z-(k-1)H))=0. \tag{8.10}$$

In fact, (8.10) implies that $H^1(N_{X'|_Z}(K_Z+D-(k-1)H))=0$ and that the restriction map

$$H^0(N_{X'|_Z}(K_Z+D-(k-1)H))\to H^0(N_{X'|_D}\otimes\omega_{X'}(-k+1))$$

is surjective. But then (8.8) and (8.9) imply (8.7).

Thus it remains to prove (8.9) and (8.10).

To this end, observe that, since Y and Z are transversal, the inclusion $T_Z\to T_{\mathbb{P}^{c+1}}$ induces a non-zero morphism $T_Z\to N_Y$, hence a zero-dimensional subscheme $\Delta_Z\subset\mathbb{P}(N_Y)$, which is mapped injectively on Y. Now (i) and (ii) imply that Δ_Z is a general finite subset of $\mathbb{P}(N_Y)$. Next, Lemma 8.4(iii) (see also [HH, Cor. 3.2]) and the exact sequence

$$0\to N_Y\to N_{X'|_Y}\to T^1_{X'}\to 0$$

give that $N^*_{X'|_Y} \cong \mathrm{elm}^-_{\Delta_Z} N^*_Y$ and therefore Lemma 8.4(i) implies that

$$N^*_{X'|_Y}((k-1)H-D) \cong (\mathrm{elm}^-_{\Delta_Z} N^*_Y)((k-1)H-D) \cong \mathrm{elm}^-_{\Delta_Z}(N^*_Y((k-1)H-D)). \tag{8.11}$$

On the other hand, since Δ_Z is general, we have by Lemma 8.4(ii) and (8.6) that

$$h^1(N_{X'|_Y}(K_Y+D-(k-1)H)) = h^0(N^*_{X'|_Y}((k-1)H-D)) = h^0(N^*_Y((k-1)H-D)) - s =$$
$$= h^1(N_Y(K_Y+D-(k-1)H)) - s = 0$$

showing (8.9). Similarly, if $\Delta_Y \subset \mathbb{P}(N_Z)$ is the finite set defined by Y, (i) and (ii) imply that Δ_Y is a general finite subset of $\mathbb{P}(N_Z)$. As above

$$N^*_{X'|_Z}((k-1)H) \cong \mathrm{elm}^-_{\Delta_Y}(N^*_Z((k-1)H))$$

and again Lemma 8.4(ii) and (8.5) give that

$$h^1(N_{X'|_Z}(K_Z-(k-1)H)) = h^1(N_Z(K_Z-(k-1)H)) - s = 0$$

showing (8.10). This concludes the proof. □

In order to find curves passing through general points with general tangent lines, we borrow an argument of Kleppe.

Lemma 8.6. *Let $P_1, \dots, P_s \in \mathbb{P}^{c+1}$ be general points and, for each $i \in \{1, \dots, s\}$, let L_i be a line in $\mathbb{P}^{c+1}$, general among the ones passing through P_i. Let $a \geq 1$ be such that*

$$(c+2)a + c - 2 - 2cs \geq 0. \tag{8.12}$$

Then there exists a rational curve $\Gamma \subset \mathbb{P}^{c+1}$ of degree a such that $P_i \in \Gamma$ and $T_{P_i}\Gamma = L_i$, for all $1 \leq i \leq s$.

Proof. It follows by [S, Prop. 2] (or by [ALY, Cor. 1.4]) and by (8.12), that the variety of rational curves of degree a in $\mathbb{P}^{c+1}$ passing through $P_1, \dots, P_s$, is non-empty. Let $\varepsilon : \widetilde{\mathbb{P}^{c+1}} \to \mathbb{P}^{c+1}$ be the blow-up of $\mathbb{P}^{c+1}$ along the points P_i, with exceptional divisors $E_i, 1 \leq i \leq s$. For any rational curve of degree a in $\mathbb{P}^{c+1}$ passing through $P_1, \dots, P_s$, the strict transform is represented by a point of the Hilbert scheme $\mathcal{H}$ of 1-dimensional closed subschemes of $\widetilde{\mathbb{P}^{c+1}}$ with Hilbert polynomial $q(t) = (ba-s)t+1$ (with respect to $\varepsilon^*\mathcal{O}_{\mathbb{P}^{c+1}}(b) - E_1 - \dots - E_s, b \gg 0$). Let $\mathcal{K}$ be the Hilbert scheme of s-tuples of points in $\widetilde{\mathbb{P}^{c+1}}$, that is with Hilbert polynomial $p(t) = s$.

Consider the flag Hilbert scheme $D(p,q)$ of Hilbert polynomials p, q, that is representing pairs $([Q_1, \dots, Q_s], [\widetilde{\Gamma}]) \in \mathcal{K} \times \mathcal{H}$ such that $Q_i \in \widetilde{\Gamma}, 1 \leq i \leq s$ and let

$$f : D(p,q) \to \mathcal{K}$$

be the forgetful morphism. By Kleppe's theorem (see [P, Thm. 1.5 and Cor. 1.6]), given a point $([Q_1, \dots, Q_s], [\widetilde{\Gamma}]) \in D(p,q)$, if $[\widetilde{\Gamma}]$ is a smooth point of $\mathcal{H}$ and if the map

$$r : H^0(N_{\widetilde{\Gamma}/\widetilde{\mathbb{P}^{c+1}}}) \to H^0(N_{\widetilde{\Gamma}/\widetilde{\mathbb{P}^{c+1}}}{}_{|\{Q_1,\dots,Q_s\}})$$

is surjective, then $\mathrm{Im} f$ contains an open neighborhood of $(Q_1, \dots, Q_s)$.

Let Γ be a rational curve of degree a in $\mathbb{P}^{c+1}$ passing through $P_1, \dots, P_s$, let $\widetilde{\Gamma} \subset \widetilde{\mathbb{P}^{c+1}}$ be the strict transform of Γ and let $Q_i \in E_i \cap \widetilde{\Gamma}, 1 \leq i \leq s$, be the points corresponding to $T_{P_i}\Gamma$. Let $\delta = \lfloor \frac{2a-2}{c} \rfloor$ and let $\rho = 2a - 2 - c\delta$. Then

$$N_{\Gamma/\mathbb{P}^{c+1}} \cong \mathcal{O}_{\mathbb{P}^1}(a+\delta)^{\oplus(c-\rho)} \oplus \mathcal{O}_{\mathbb{P}^1}(a+\delta+1)^{\oplus\rho}$$

by [S, Prop. 2]. Also, [ACPS, Lemma 2.1] (see also [CS, Rmk. 4.2.7]) implies that

$$N_{\widetilde{\Gamma}/\widetilde{\mathbb{P}^{c+1}}} \cong N_{\Gamma/\mathbb{P}^{c+1}}(-s) \cong \mathcal{O}_{\mathbb{P}^1}(a+\delta-s)^{\oplus(c-\rho)} \oplus \mathcal{O}_{\mathbb{P}^1}(a+\delta+1-s)^{\oplus\rho}. \tag{8.13}$$

But then (8.12) and (8.13) give that

$$H^1(N_{\widetilde{\Gamma}/\widetilde{\mathbb{P}^{c+1}}}(-Q_1 - \dots - Q_s)) = H^1(\mathcal{O}_{\mathbb{P}^1}(a+\delta-2s)^{\oplus(c-\rho)} \oplus \mathcal{O}_{\mathbb{P}^1}(a+\delta+1-2s)^{\oplus\rho}) = 0$$

Hence r is surjective and $H^1(N_{\widetilde{\Gamma}/\widetilde{\mathbb{P}^{c+1}}}) = 0$, so that $[\widetilde{\Gamma}]$ is a smooth point of $\mathcal{H}$ and $\mathrm{Im} f$ contains an open neighborhood of $(Q_1, \dots, Q_s)$. Therefore we can find a curve $\widetilde{\Gamma}' \subset \widetilde{\mathbb{P}^{c+1}}$, represented by a point of $\mathcal{H}$, passing through s general points $Q'_i \in E_i, 1 \le i \le s$. Finally $\Gamma' := f(\widetilde{\Gamma}')$ is a rational curve of degree a in $\mathbb{P}^{c+1}$ such that $P_i \in \Gamma'$ and with tangent lines at the P_i's that are general lines among the ones passing through the P_i's, corresponding to the s general points $Q'_i \in E_i, 1 \le i \le s$. □

Remark 8.7. Let $\Gamma \subset \mathbb{P}^{c+1}$ be a general nonspecial curve of degree a and genus γ, let $P_1, \dots, P_s \in \Gamma$ be general points and assume that $(c+2)a - (c-2)(\gamma - 1) - 2cs \ge 0$ and that

$$H^1(N_{\Gamma/\mathbb{P}^{c+1}}(-2P_1 - \ldots - 2P_s)) = 0. \tag{8.14}$$

Then, with the same proof of Lemma 8.6, we can find a nonspecial curve of degree a and genus γ, passing through s general points with s general tangent lines at these points. We do not know if (8.14) holds, but if so, most likely several more examples of curves having $N_X^*(k)$ Ulrich could be found.

We will now apply Lemma 8.5 to general nonspecial curves. We first prove some lemmas that allow the construction.

As in [HH] (but we allow disconnectedness) we use the following

Definition 8.8. A *stick figure* is a reduced nodal one-dimensional closed subscheme in $\mathbb{P}^r$, whose irreducible components are lines.

We will use the next lemma, often for stick figures.

Lemma 8.9. *Let $\Gamma, \Gamma' \subset \mathbb{P}^r$ be two reduced nodal one-dimensional closed subschemes without common components and such that $\Gamma \cup \Gamma'$ is nodal and:*

(i) *Γ is nonspecial.*
(ii) $H^1(\mathcal{O}_{\Gamma'}(1)(-\Gamma \cap \Gamma')) = 0$.

Then also $\Gamma \cup \Gamma'$ is nonspecial.

Proof. The statement follows by (i), (ii) and the exact sequence

$$0 \to \mathcal{O}_{\Gamma'}(1)(-\Gamma \cap \Gamma') \to \mathcal{O}_{\Gamma \cup \Gamma'}(1) \to \mathcal{O}_{\Gamma}(1) \to 0.$$

□

We will now prove four lemmas, all in the following setting: let

$$P_1, \cdots P_{2b} \in \mathbb{P}^3 \text{ be general points.}$$

For each $i \in \{1, \dots, 2b\}$, let L_i be a line in $\mathbb{P}^3$, general among the ones passing through P_i.

Lemma 8.10. *If $b \ge 2$, there are lines $M_1, \dots, M_b$ and a connected nonspecial stick figure*

$$Z_b = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_b$$

having P_i's as smooth points and such that $p_a(Z_b) = b + 1$.

Proof. The lemma will follow by Lemma 8.9 and the following

Claim 8.11. *There is a nonspecial stick figure $M_1 \cup \ldots \cup M_b$ such that Z_b is a connected stick figure, $p_a(Z_b) = b+1$, the P_i's are smooth points of Z_b and*

$$H^1(\mathcal{O}_{L_1 \cup \ldots \cup L_{2b}}(1)(-B_b)) = 0, \text{ where } B_b = (L_1 \cup \ldots \cup L_{2b}) \cap (M_1 \cup \ldots \cup M_b). \tag{8.15}$$

We now prove Claim 8.11 by induction on b.

If $b = 2$, let Q be the smooth quadric containing L_1, L_2 and L_3. Then L_4 meets Q at the two points R_1, R_2. Take two lines M_1, M_2 of the other ruling of Q (with respect to L_1) passing through R_1 and R_2 respectively and let (see Figure 1)

$$Z_2 = L_1 \cup \ldots \cup L_4 \cup M_1 \cup M_2.$$

Then the P_i's are smooth points of Z_2 and $p_a(Z_2) = 3$. Since M_1 and M_2 are disjoint, we have that $M_1 \cup M_2$ is nonspecial. Moreover

$$H^1(\mathcal{O}_{L_1 \cup L_2 \cup L_3 \cup L_4}(1)(-B_2)) = H^1(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 4}) = 0 \tag{8.16}$$

and this gives the case $b = 2$.

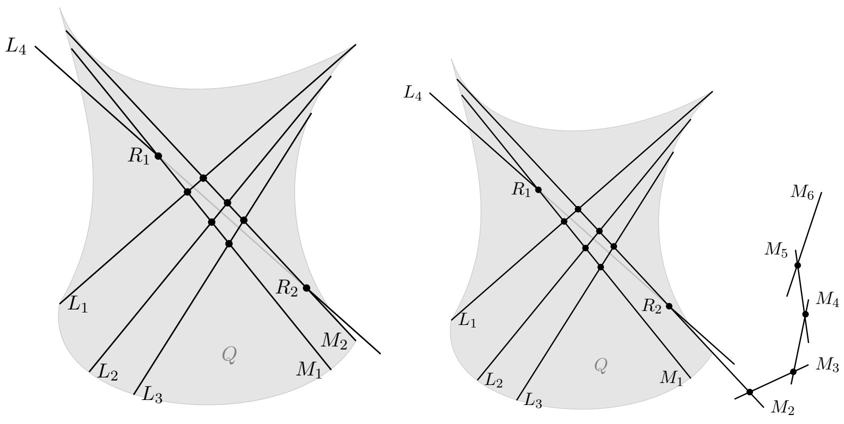

FIGURE 1. Left: $p_a(Z_2) = 3, d(Z_2) = 6$. Right: $p_a(Y_2) = 3, d(Y_2) = 10$

Now, if $b \geq 3$, by induction we have a nonspecial stick figure $M_1 \cup \ldots \cup M_{b-1}$ such that Z_{b-1} is a connected stick figure, $p_a(Z_{b-1}) = b$, the P_i's are smooth points of Z_{b-1} and

$$H^1(\mathcal{O}_{L_1\cup\ldots\cup L_{2b-2}}(1)(-B_{b-1})) = 0. \tag{8.17}$$

In particular, $\deg(B_{b-1|L_i}) \leq 2$ for all $1 \leq i \leq 2b-2$. Let Q be the smooth quadric containing M_{b-1}, L_{2b-1} and L_{2b} and let R_0 be the point of intersection of Q with M_{b-2}, not lying on M_{b-1}. Let M_b be the ruling of Q of the other kind passing through R_0 (with respect to M_{b-1}) intersecting $M_{b-1}, L_{2b-1}, L_{2b}$ at R_1, R_2, R_3 respectively. Then $M_1 \cup \ldots \cup M_b$ is a nonspecial stick figure by Lemma 8.9. Moreover, if

$$Z_b = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_b$$

then Z_b has as nodes the nodes of Z_{b-1} and R_0, R_1, R_2, R_3, whence $p_a(Z_b) = b+1$. Now we note that $B_{b|L_i} = B_{b-1|L_i}$ for $1 \leq i \leq 2b-2$, $B_{b|L_{2b-1}} = R_2$ and $B_{b|L_{2b}} = R_3$. Hence (8.17) gives (8.15). □

Lemma 8.12. *If $b \geq 2$, there are lines $M_1, \ldots, M_{b+4}$ and a connected nonspecial stick figure*

$$Y_b = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_{b+4}$$

having P_i's as smooth points and such that $p_a(Y_b) = 3$.

Proof. The lemma will follow by Lemma 8.9 and the following

Claim 8.13. *There is a nonspecial stick figure $M_1 \cup \ldots \cup M_{b+4}$ such that Y_b is a connected stick figure, $p_a(Y_b) = 3$, the P_i's are smooth points of Y_b and*

$$H^1(\mathcal{O}_{L_1\cup\ldots\cup L_{2b}}(1)(-B_b)) = 0, \ \textit{where } B_b = (L_1 \cup \ldots \cup L_{2b}) \cap (M_1 \cup \ldots \cup M_{b+4}). \tag{8.18}$$

We now prove Claim 8.13 by induction on b.

If $b = 2$, let $Z_2 = L_1 \cup \ldots \cup L_4 \cup M_1 \cup M_2$ be the stick figure constructed in Lemma 8.10 for $b = 2$ and let $B_2' = (L_1 \cup \ldots \cup L_4) \cap (M_1 \cup M_2)$. For $3 \leq i \leq 6$, let M_i be a general line meeting only M_{i-1} in one point and set (see Figure 1)

$$Y_2 = L_1 \cup \ldots L_4 \cup M_1 \cup \ldots \cup M_6.$$

Then $M_1 \cup \ldots \cup M_6$ is a nonspecial stick figure by Lemma 8.9. Moreover, $p_a(Y_2) = p_a(Z_2) = 3$ and $B_2 = B_2'$, hence (8.18) holds by (8.15).

Now, if $b \geq 3$, by induction we have a nonspecial stick figure $M_1 \cup \ldots \cup M_{b+3}$ such that Y_{b-1} is a connected stick figure, $p_a(Y_{b-1}) = 3$, the P_i's are smooth points of Y_{b-1} and

$$H^1(\mathcal{O}_{L_1\cup\ldots\cup L_{2b-2}}(1)(-B_{b-1})) = 0. \tag{8.19}$$

In particular, $\deg(B_{b-1|L_i}) \leq 2$ for all $1 \leq i \leq 2b-2$. Let Q be a quadric containing $M_{b+3}, L_{2b-1}, L_{2b}$ and let M_{b+4} be a general ruling of Q of the other kind (with respect to M_{b+3}) intersecting $M_{b+3}, L_{2b-1}, L_{2b}$ at R_1, R_2, R_3 respectively. Then $M_1 \cup \ldots \cup M_{b+4}$ is a nonspecial stick figure by Lemma 8.9. Moreover, if

$$Y_b = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_{b+4}$$

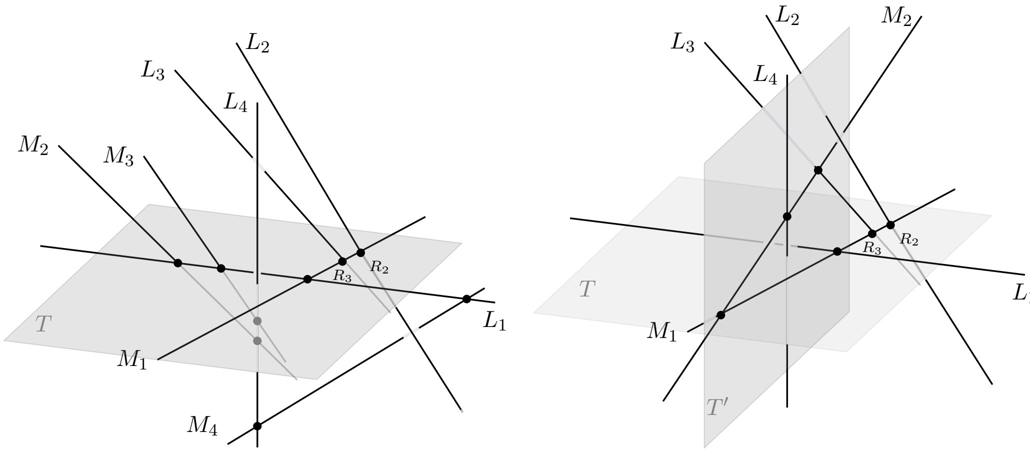

FIGURE 2. Left: $p_a(Y_2') = 2, d(Y_2') = 8$. Right: $p_a(Y_2'') = 1, d(Y_2'') = 6$.

then Y_b has as nodes the nodes of Y_{b-1} and R_1, R_2, R_3, whence $p_a(Y_b) = 3$. Now we note that $B_{b|L_i} = B_{b-1|L_i}$ for $1 \leq i \leq 2b-2$, $B_{b|L_{2b-1}} = R_2$ and $B_{b|L_{2b}} = R_3$. Hence (8.19) gives (8.18). □

Lemma 8.14. *If $b \geq 2$, there are disjoint lines $M_1, \ldots, M_{b+2}$ and a connected nonspecial stick figure*

$$Y_b' = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_{b+2}$$

having P_i's as smooth points and such that $p_a(Y_b') = 2$.

Proof. We prove the lemma by induction on b.

If $b = 2$, let T be a plane containing L_1. Let R_2 (respectively R_3) be the point of intersection of L_2 (respectively L_3) with T. Let M_1 be a line joining R_2 and R_3, intersecting L_1 at R_1. Further, let M_2, M_3, M_4 be three general 2-secant lines to $L_1 \cup L_4$ and set (see Figure 2)

$$Y_2' = L_1 \cup \ldots L_4 \cup M_1 \cup \ldots \cup M_4$$

so that $p_a(Y_2') = 2$. Now $L_1 \cup M_1$ is a conic, hence $H^1(\mathcal{O}_{L_1 \cup M_1}(1)) = 0$ and then also $H^1(\mathcal{O}_{L_1 \cup L_4 \cup M_1}(1)) = H^1(\mathcal{O}_{L_1 \cup M_1}(1) \oplus \mathcal{O}_{L_4}(1)) = 0$. Moreover, setting $B = (L_2 \cup L_3 \cup M_2 \cup M_3 \cup M_4) \cap (L_1 \cup L_4 \cup M_1)$, we see that $\mathcal{O}_{L_2 \cup L_3 \cup M_2 \cup M_3 \cup M_4}(1)(-B) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 3}$, hence $H^1(\mathcal{O}_{L_2 \cup L_3 \cup M_2 \cup M_3 \cup M_4}(1)(-B)) = 0$. Therefore Y_2' is nonspecial by Lemma 8.9 and this gives the case $b = 2$.

Now, if $b \geq 3$, by induction there are disjoint lines $M_1, \ldots, M_{b+1}$ and a connected nonspecial stick figure

$$Y_{b-1}' = L_1 \cup \ldots \cup L_{2b-2} \cup M_1 \cup \ldots \cup M_{b+1}$$

having P_i's as smooth points and such that $p_a(Y_{b-1}') = 2$. Let Q be the quadric containing L_{2b-2}, L_{2b-1} and L_{2b}. Let M_{b+2} be a general ruling of Q of the other kind (with respect to L_{2b-2}). Now, if

$$Y_b' = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_{b+2}$$

we have that $p_a(Y_b') = 2$. Moreover, setting $B' = (L_{2b-1} \cup L_{2b} \cup M_{b+2}) \cap Y_{b-1}'$, we see that $\mathcal{O}_{L_{2b-1} \cup L_{2b} \cup M_{b+2}}(1)(-B') \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$, hence $H^1(\mathcal{O}_{L_{2b-1} \cup L_{2b} \cup M_{b+2}}(1)(-B')) = 0$. Therefore Y_b' is nonspecial by Lemma 8.9. □

Lemma 8.15. *If $b \geq 2$, there are lines $M_1, \ldots, M_b$ and a connected nonspecial stick figure*

$$Y_b'' = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_b$$

having P_i's as smooth points and such that $p_a(Y_b'') = 1$.

Proof. The lemma will follow by Lemma 8.9 and the following

Claim 8.16. *There is a nonspecial stick figure $M_1 \cup \ldots \cup M_b$ such that Y_b'' is a connected stick figure, $p_a(Y_b'') = 1$, the P_i's are smooth points of Y_b'' and*

(8.20) $H^1(\mathcal{O}_{L_1 \cup \ldots \cup L_{2b}}(1)(-B_b)) = 0, \deg(B_{b|L_{2b}}) = 1$ *where* $B_b = (L_1 \cup \ldots \cup L_{2b}) \cap (M_1 \cup \ldots \cup M_b)$.

We now prove Claim 8.16 by induction on b.

If $b = 2$, pick a general plane T containing L_1. Then T meets L_2 in a point R_2 and L_3 in a point R_3. Let M_1 be the line joining R_2 and R_3. Then M_1 meets L_1 in one point. Now take a general plane T' containing L_4. Then T' meets M_1 and L_3 in two distinct points and if M_2 is the line joining them, then (see Figure 2)

$$Y_2'' = L_1 \cup L_2 \cup L_3 \cup L_4 \cup M_1 \cup M_2$$

is such that $p_a(Y_2'') = 1$. Now $M_1 \cup M_2$ is a conic, hence is nonspecial and nodal. Moreover

$$\mathcal{O}_{L_1\cup L_2 \cup L_3 \cup L_4}(1)(-B_2) \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$$

hence (8.20) holds and this gives the case $b = 2$.

Now, if $b \geq 3$, by induction we have a nonspecial stick figure $M_1 \cup \ldots \cup M_{b-1}$ such that Y_{b-1}'' is a connected stick figure, $p_a(Y_{b-1}'') = 1$, the P_i's are smooth points of Y_{b-1}'' and

$$H^1(\mathcal{O}_{L_1\cup \ldots \cup L_{2b-2}}(1)(-B_{b-1})) = 0 \text{ and } \deg(B_{b-1|L_{2b-2}}) = 1. \tag{8.21}$$

In particular, $\deg(B_{b-1|L_i}) \leq 2$ for all $1 \leq i \leq 2b-3$. Let Q be the quadric containing $L_{2b-2}, L_{2b-1}, L_{2b}$ and let M_b be a general ruling of Q of the other kind (with respect to L_{2b-2}), intersecting $L_{2b-2}, L_{2b-1}, L_{2b}$ at R_1, R_2, R_3, respectively. Then $M_1 \cup \ldots \cup M_b$ is a nonspecial stick figure and if

$$Y_b'' = L_1 \cup \ldots \cup L_{2b} \cup M_1 \cup \ldots \cup M_b$$

then Y_b'' has as nodes the nodes of Y_{b-1}'' and R_1, R_2, R_3, whence $p_a(Y_b'') = 1$. Now we note that $B_{b|L_i} = B_{b-1|L_i}$ for $1 \leq i \leq 2b-3, \deg(B_{b|L_{2b-2}}) = 2, B_{b|L_{2b-1}} = R_2$ and $B_{b|L_{2b}} = R_3$. Hence (8.21) gives (8.20). □

Next, we prove Theorem 3. We will often use the fact, proved for example in [HH, Cor. 1.2], that a nonspecial nodal reduced one-dimensional closed subscheme in $\mathbb{P}^{c+1}$ is smoothable.

Proof of Theorem 3(i). From $d = 2g - 2$ we deduce that $g \geq 2$, hence, since X is nonspecial, $g - 1 = h^0(\mathcal{O}_C(1)) \geq 3$. Thus $g \geq 4$ and if equality holds, then C is a plane curve of degree 6, a contradiction. Therefore $g \geq 5$.

Let $P_1, \cdots P_4 \in \mathbb{P}^3$ be general points and, for each $i \in \{1, \ldots, 4\}$, let L_i, L_i' be lines in $\mathbb{P}^3$, general among the ones passing through P_i.

We first consider the case $g = 6$.

Claim 8.17. *There are a smooth rational quartic $Y \subset \mathbb{P}^3$, a smooth nonspecial curve $Z \subset \mathbb{P}^3$ of degree 6 and genus 3, such that:*

(i) $D = Y \cap Z$ *is a general effective divisor of degree* 4 *on both.*
(ii) $T_{P_i}Y = L_i, T_{P_i}Z = L_i', 1 \leq i \leq 4.$
(iii) $H^0(N_Z(K_Z - 3H)) = H^0(N_Y(K_Y + D - 3H)) = 0.$
(iv) $X' = Y \cup Z$ *is nonspecial.*

Proof of Theorem 3(i). The existence of Y passing through $P_1, \cdots P_4$ and satisfying (ii) is assured by Lemma 8.6. As for (iii), observe that $N_Y \cong \mathcal{O}_{\mathbb{P}^1}(7)^{\oplus 2}$ by [S, Prop. 2], hence (iii) holds.

Next, to find Z, we argue by degeneration. Let $Z' = Z_2$ be the nonspecial stick figure of degree 6 and arithmetic genus 3 constructed in Claim 8.11 (now using $L_1', \ldots, L_4'$). Then Z' is smoothable, $Y \cap Z' = D$ and $p_a(Y \cup Z') = 6$. Hence any smoothing $Y \cup Z$ obtained by smoothing Z' so that it still passes through 4 general points, must have $p_a(Y \cup Z) = 6$. That is, again $Y \cap Z = D$, meeting transversally and with two different sets of tangent lines at the 4 points that are general lines among the ones passing through these points. Next, X' is nonspecial by Lemma 8.9, since $H^1(\mathcal{O}_Z(1)) = 0$ and $H^1(\mathcal{O}_Y(1)(-D)) = H^1(\mathcal{O}_{\mathbb{P}^1}) = 0$. Thus it remains to check (iii) for a general nonspecial curve $Z \subset \mathbb{P}^3$ of degree 6 and genus 3. To this end, we specialize Z to a curve Z'' of type $(2, 4)$ on a smooth quadric $Q \subset \mathbb{P}^3$. Observe that the exact sequence

$$0 \to \mathcal{O}_Q(-1, -3) \to \mathcal{O}_Q(1) \to \mathcal{O}_{Z''}(1) \to 0$$

shows that $H^1(\mathcal{O}_{Z''}(1)) = 0$, hence $H^0(\mathcal{O}_{Z''}(K_{Z''} - H)) = 0$ by Serre's duality. Also, the exact sequence

$$0 \to \mathcal{O}_Q(-3, -1) \to \mathcal{O}_Q(-1, 3) \to \mathcal{O}_{Z''}(Z'' + K_{Z''} - 3H) \to 0$$

shows that $H^0(\mathcal{O}_{Z''}(Z''+K_{Z''}-3H))=0$. Therefore the exact sequence

$$0 \to \mathcal{O}_{Z''}(Z''+K_{Z''}-3H) \to N_{Z''}(K_{Z''}-3H) \to \mathcal{O}_{Z''}(K_{Z''}-H) \to 0$$

implies that $H^0(N_{Z''}(K_{Z''}-3H))=0$. □

Next assume that $g \ge 7$. Set $g=3b+\varepsilon$, with $1 \le \varepsilon \le 3$ and let $s=2b$. Let $P_1,\dots,P_s \in \mathbb{P}^3$ be general points and set $D=\{P_1,\dots,P_s\}$. For each $i \in \{1,\dots,s\}$, let L_i, L_i' be lines in $\mathbb{P}^3$, general among the ones passing through P_i.

We first construct the appropriate degenerations according to ε.

Claim 8.18. *If $\varepsilon=3$ there are a smooth curve $Y \subset \mathbb{P}^3$ of degree $g+1$ and genus 3, a smooth curve $Z \subset \mathbb{P}^3$ of degree $g-3$ and genus $\frac{g}{3}$, such that:*

(i) *$D=Y\cap Z$ is a general effective divisor of degree s on both.*
(ii) *$T_{P_i}Y=L_i, T_{P_i}Z=L_i', 1 \le i \le s$.*
(iii) *N_Y and N_Z are stable.*
(iv) *$X'=Y \cup Z$ is nonspecial.*

Proof. It follows by [ALY, Thm. 1.3] that there are a general nonspecial curve $Y \subset \mathbb{P}^3$ of degree $g+1$ and genus 3, and a general nonspecial curve $Z \subset \mathbb{P}^3$ of degree $g-3$ and genus $\frac{g}{3}$, both containing D and satisfying $H^1(N_Y(-D))=H^1(N_Z(-D))=0$. Moreover, by Kleppe's theorem (see for example [P, Thm. 1.5] or [ALY, Thm. 1.1]), the generality assumptions give that D is a general effective divisor on both.

To show that these curves meet only along D with the assigned tangent lines, we will argue by degeneration.

Let $Y'=Y_b$ be the stick figure in Lemma 8.12 and let $Z'=Z_b$ be the stick figure in Lemma 8.10. Since they are both nonspecial, they are smoothable. Observe that $Y' \cap Z'=D$ and $p_a(Y'\cup Z')=g$. Hence any smoothing $Y \cup Z$ obtained by smoothing Y' and Z' so that they still pass through $2b$ general points, must have $p_a(Y\cup Z)=g$. That is, again $Y\cap Z=D$, meeting transversally and with two different sets of tangent lines at the $2b$ points that are general lines passing through these points. Next, N_Y and N_Z are stable by [CLV, Main Thm.]. Finally, X' is nonspecial by Lemma 8.9, since $H^1(\mathcal{O}_Z(1))=0$ and $H^1(\mathcal{O}_Y(1)(-D))=0$, since $d_Y-s>2g_Y-2$. □

Claim 8.19. *If $\varepsilon=1$ there are a smooth curve $Y \subset \mathbb{P}^3$ of degree $g-1$ and genus 1, a smooth curve $Z \subset \mathbb{P}^3$ of degree $g-1$ and genus $\frac{g+2}{3}$, such that:*

(i) *$D=Y\cap Z$ is a general effective divisor of degree s on both.*
(ii) *$T_{P_i}Y=L_i, T_{P_i}Z=L_i', 1 \le i \le s$.*
(iii) *N_Y is semistable and N_Z is stable.*
(iv) *$X'=Y \cup Z$ is nonspecial.*

Proof. The proof is the same as the one of Claim 8.18 except that we now choose $Y'=Y_b''$ be the stick figure in Lemma 8.15. □

Claim 8.20. *If $\varepsilon=2$ there are a smooth curve $Y \subset \mathbb{P}^3$ of degree g and genus 2, a smooth curve $Z \subset \mathbb{P}^3$ of degree $g-2$ and genus $\frac{g+1}{3}$, such that:*

(i) *$D=Y\cap Z$ is a general effective divisor of degree s on both.*
(ii) *$T_{P_i}Y=L_i, T_{P_i}Z=L_i', 1 \le i \le s$.*
(iii) *N_Y and N_Z are stable.*
(iv) *$X'=Y \cup Z$ is nonspecial.*

Proof. The proof is the same as the one of Claim 8.18 except that we now choose $Y'=Y_b'$ be the stick figure in Lemma 8.14. □

We now conclude the proof of Theorem 3(i).

For the cases $g=5,6$ we consider $P_1,\cdots P_4 \in \mathbb{P}^3$ general points and, for each $i \in \{1,\dots,4\}$, L_i, L_i' be lines in $\mathbb{P}^3$, general among the ones passing through P_i.

First, we deal with the case $g=5$.

Let $Z \subset \mathbb{P}^3$ be the curve of degree 6 and genus 3 constructed in Claim 8.17. Let Y_1 (respectively Y_2) be the line joining P_1 and P_2 (resp. P_3 and P_4). Since Z is nonspecial, it follows by Lemma 8.9 that also

$$X' = Z \cup Y_1 \cup Y_2$$

is nonspecial. Moreover, for $j = 1, 2$, setting $D_1 = P_1 + P_2, D_2 = P_3 + P_4$, we have that

$$h^1(N_{Y_j}(K_{Y_j} + D_j - 3H)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-2)^{\oplus 2}) = 2. \tag{8.22}$$

We now argue as in the proof of Lemma 8.5.

Since $T_{P_i}Z = L_i', 1 \le i \le 4$, the finite set $\Delta_j \subset \mathbb{P}(N_{Y_j}), j = 1, 2$, defined by Z, is just a general finite subset of $\mathbb{P}(N_{Y_j})$. Now, as in the proof of Lemma 8.5, we have that

$$N^*_{X'|_{Y_j}}(3H - D_j) \cong \mathrm{elm}^-_{\Delta_j}(N^*_{Y_j}(3H - D_j))$$

and then, since Δ_j is general, we have by Lemma 8.4(ii) and (8.22) that

$$h^1(N_{X'|_{Y_j}}(K_{Y_j}+D_j-3H)) = h^0(N^*_{X'|_{Y_j}}(3H-D_j)) = h^0(N^*_{Y_j}(3H-D_j))-2 = h^1(N_{Y_j}(K_{Y_j}+D_j-3H))-2 = 0. \tag{8.23}$$

Next, we show that

$$h^1(N_{X'|_Z}(K_Z - 3H)) = 0. \tag{8.24}$$

To prove this, we choose a general quadric $Q \subset \mathbb{P}^3$ passing through $P_1, \dots, P_4$ and a general curve Z'' of type $(2, 4)$ on Q still passing through $P_1, \dots, P_4$. Then we specialize X' to

$$X'' = Z'' \cup Y_1' \cup Y_2'.$$

where Y_1' (respectively Y_2') are the line joining P_1 and P_2 (resp.P_3 and P_4). Now (8.24) will follow by semicontinuity from

$$h^1(N_{X''|_{Z''}}(K_{Z''} - 3H)) = 0. \tag{8.25}$$

We already showed in the proof of Claim 8.17 that $H^0(N_{Z''}(K_{Z''} - 3H)) = 0$, hence

$$h^1(N_{Z''}(K_{Z''} - 3H)) = 4. \tag{8.26}$$

On the other hand, letting $\Delta'' \subset \mathbb{P}(N_{Z''})$ be the finite set defined by $Y_1' \cup Y_2'$, as in the proof of Lemma 8.5, we have that

$$N^*_{X''|_{Z''}}(3H) \cong \mathrm{elm}^-_{\Delta''}(N^*_{Z''}(3H)). \tag{8.27}$$

We now observe that Δ'' imposes independent conditions to $\mathcal{O}_{\mathbb{P}(N^*_{Z''}(3H))}(1)$. In fact, since Z'' lies on a quadric, we have an inclusion $H^0(H) \subseteq H^0(N^*_{Z''}(3H)) \cong H^0(\mathcal{O}_{\mathbb{P}(N^*_{Z''}(3H))}(1))$. If $\pi : \mathbb{P}(N^*_{Z''}(3H)) \to Z''$ is the projection, then $\pi(\Delta'') = \{P_1, \dots, P_4\}$. Hence, for any $j \in \{1, \dots, 4\}$, we can find a section $\sigma_j \in H^0(H) \subseteq H^0(N^*_{Z''}(3H))$ such that $\sigma_j(P_j) \neq 0, \sigma_j(P_i) = 0$ for $i \in \{1, \dots, 4\}, i \neq j$. Therefore the sections $\pi^*\sigma_j \in H^0(\mathcal{O}_{\mathbb{P}(N^*_{Z''}(3H))}(1)), 1 \le j \le 4$ show that Δ'' imposes independent conditions to $\mathcal{O}_{\mathbb{P}(N^*_{Z''}(3H))}(1)$. But then, using (8.27) and (8.26), we have

$$h^1(N_{X''|_{Z''}}(K_{Z''} - 3H)) = h^0(N^*_{X''|_{Z''}}(3H)) = H^0(\mathrm{elm}^-_{\Delta''}(N^*_{Z''}(3H))) = h^0(N^*_{Z''}(3H)) - 4 =$$

$$= h^1(N_{Z''}(K_{Z''} - 3H)) - 4 = 0.$$

This proves (8.25), hence (8.24). Now, exactly as in the proof of Lemma 8.5 we conclude (with the analogue of sequence (8.8)), from (8.23) and (8.24), that $H^1(N_{X'} \otimes \omega_{X'}(-3)) = 0$. Then a general smoothing $X \subset \mathbb{P}^3$ of X' is such that $\deg X = 8, g(X) = 5$ and $N^*_X(4)$ is Ulrich. Finally X is not subcanonical, for otherwise we would have that $K_C = H$, contradicting the fact that X is nonspecial. This concludes the proof in the case $g = 5$.

When $g = 6$, in the case of Claim 8.17, it is easily verified that the conditions (i)-(vii) (using (iii-2) and (iv-2)) of Lemma 8.5 are satisfied with $k = 4, s = 4, c = 2$.

Next, for the case $g \ge 7$, we set $g = 3b+\varepsilon$, with $1 \le \varepsilon \le 3$ and let $s = 2b$. We consider $P_1, \dots, P_s \in \mathbb{P}^3$ general points and, for each $i \in \{1, \dots, s\}$, we let L_i, L_i' be lines in $\mathbb{P}^3$, general among the ones passing through P_i.

In all three cases of Claims 8.18, 8.19 and 8.20, it is easily verified that the conditions (i)-(vii) (using (iii-1) and (iv-1)) of Lemma 8.5 are satisfied with $k = 4, s = 2b, c = 2$.

Moreover, $X' = Y \cup Z$ a nodal nonspecial curve with $\deg X' = 2g - 2$ and $p_a(X') = g$. In particular X' is smoothable in $\mathbb{P}^3$. Then a general smoothing $X \subset \mathbb{P}^3$ of X' is such that $\deg X = 2g-2, g(X) = g$ and $N_X^*(4)$ is Ulrich by Lemma 8.5. Finally X is not subcanonical, for otherwise we would have that $K_C = H$, contradicting the fact that X is nonspecial. □

We will also use the following

Remark 8.21. Let L_1, L_2, L_3 be three disjoint lines in $\mathbb{P}^4$. Then there is a trisecant line L to $L_1 \cup L_2 \cup L_3$.

Proof. Observe that $H = \langle L_1, L_2 \rangle$ is a hyperplane in $\mathbb{P}^4$, hence H meets L_3 in a point P_3. Then $\{P_3\} \cup L_1 \cup L_2 \subset H \cong \mathbb{P}^3$ and if $M = \langle P_3, L_1 \rangle$, then M is a plane in H, meeting L_2 in a point P_2. Therefore the line L joining P_2 and P_3 lies in M, hence meets L_1. □

Finally, we prove Theorem 3(ii).

Proof of Theorem 3(ii). Write $g = 29 + 4b + \epsilon$ with $b \geq 0, 0 \leq \epsilon \leq 3$ and set

$$s = 22 + 3b + \epsilon.$$

Let $P_1, \dots, P_s \in \mathbb{P}^4$ be general points and set $D = \{P_1, \dots, P_s\}$. For each $i \in \{1, \dots, s\}$, let L_i, L_i' be lines in $\mathbb{P}^4$, general among the ones passing through P_i.

Claim 8.22. *There is a smooth rational curve $Y \subset \mathbb{P}^4$ of degree $83 + 12b + 4\epsilon$ and a smooth curve $Z \subset \mathbb{P}^4$ of degree $57 + 8b + \epsilon$ and genus $8 + b$, such that:*

(i) *$D = Y \cap Z$ is a general effective divisor of degree s on both.*
(ii) *$T_{P_i}Y = L_i, T_{P_i}Z = L_i', 1 \leq i \leq s$.*
(iii) *N_Z is stable.*
(iv) *$X' = Y \cup Z$ is nonspecial.*

Proof. By Lemma 8.6 we can find a smooth rational curve $Y \subset \mathbb{P}^4$ of degree $83 + 12b + 4\epsilon$ passing through $P_1, \dots, P_s$ with tangent lines at the P_i's that are general lines among the ones passing through these points.

Next, to find Z, we argue by degeneration, considering different cases according to ϵ.

<u>Case 0: $\epsilon = 0$.</u> Fix i with $0 \leq i \leq 6 + b$. Let M'_{2i+1} be a trisecant line to $L'_{3i+1} \cup L'_{3i+2} \cup L'_{3i+3}$, that exists by Remark 8.21. Similarly let M'_{2i+2} be a trisecant line to $L'_{3i+2} \cup L'_{3i+3} \cup L'_{3i+4}$. Let M'_{15+2b} be a general chord of $L'_1 \cup L'_{22+3b}$. Set $B' = (L'_1 \cup \ldots \cup L'_{22+3b}) \cap (M'_1 \cup \ldots \cup M'_{15+2b})$ and observe that $\mathcal{O}_{L'_1 \cup \ldots \cup L'_{22+3b}}(1)(-B') \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(22+3b)}$. Thus, Lemma 8.9 gives that $L'_1 \cup \ldots \cup L'_{22+3b} \cup M'_1 \cup \ldots \cup M'_{15+2b}$ is nonspecial. We now take a tree of lines $M'_{16+2b} \cup \ldots \cup M'_{35+5b}$ such that M'_i intersects M'_{i-1} at one point, for $16 + 2b \leq i \leq 35 + 5b$. Set

$$Z' = L'_1 \cup \ldots \cup L'_{22+3b} \cup M'_1 \cup \ldots \cup M'_{35+5b}$$

which is a connected stick figure of degree $57 + 8b$ and genus $8 + b$, and it is nonspecial by Lemma 8.9.

<u>Case 1: $\epsilon = 1$.</u> Fix i with $0 \leq i \leq 6 + b$. Let M'_{2i+1} be a trisecant line to $L'_{3i+1} \cup L'_{3i+2} \cup L'_{3i+3}$, that exists by Remark 8.21. Similarly let M'_{2i+2} be a trisecant line to $L'_{3i+2} \cup L'_{3i+3} \cup L'_{3i+4}$. Let M'_{15+2b} be a trisecant line of $L'_1 \cup L'_{22+3b} \cup L'_{23+3b}$. Set $B' = (L'_1 \cup \ldots \cup L'_{23+3b}) \cap (M'_1 \cup \ldots \cup M'_{15+2b})$ and observe that $\mathcal{O}_{L'_1 \cup \ldots \cup L'_{23+3b}}(1)(-B') \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(22+3b)}$. Thus, Lemma 8.9 gives that $L'_1 \cup \ldots \cup L'_{23+3b} \cup M'_1 \cup \ldots \cup M'_{15+2b}$ is nonspecial. We now take a tree of lines $M'_{16+2b} \cup \ldots \cup M'_{35+5b}$ such that M'_i intersects M'_{i-1} at one point, for $16 + 2b \leq i \leq 35 + 5b$. Set

$$Z' = L'_1 \cup \ldots \cup L'_{23+3b} \cup M'_1 \cup \ldots \cup M'_{35+5b}$$

which is a connected stick figure of degree $58 + 8b$ and genus $8 + b$, and it is nonspecial by Lemma 8.9.

<u>Case 2: $\epsilon = 2$.</u> For i with $0 \leq i \leq 7+b$, let M'_{2i+1} be a trisecant line to $L'_{3i+1} \cup L'_{3i+2} \cup L'_{3i+3}$, that exists by Remark 8.21. Similarly, for i with $0 \leq i \leq 6+b$, let M'_{2i+2} be a trisecant line to $L'_{3i+2} \cup L'_{3i+3} \cup L'_{3i+4}$. Let M'_{16+2b} be a general chord of $L'_{23+3b} \cup L'_{24+3b}$. Set $B' = (L'_1 \cup \ldots \cup L'_{24+3b}) \cap (M'_1 \cup \ldots \cup M'_{16+2b})$ and observe that $\mathcal{O}_{L'_1 \cup \ldots \cup L'_{24+3b}}(1)(-B') \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(23+3b)}$. Thus, Lemma 8.9 gives that $L'_1 \cup$

$\ldots \cup L'_{24+3b} \cup M'_1 \cup \ldots \cup M'_{16+2b}$ is nonspecial. We now take a tree of lines $M'_{17+2b} \cup \ldots \cup M'_{35+5b}$ such that M'_i intersects M'_{i-1} at one point, for $17+2b \le i \le 35+5b$. Set

$$Z' = L'_1 \cup \ldots \cup L'_{24+3b} \cup M'_1 \cup \ldots \cup M'_{35+5b}$$

which is a connected stick figure of degree $59+8b$ and genus $8+b$, and it is nonspecial by Lemma 8.9.

Case 3: $\epsilon = 3$. Fix i with $0 \le i \le 7+b$. Let M'_{2i+1} be a trisecant line to $L'_{3i+1} \cup L'_{3i+2} \cup L'_{3i+3}$, that exists by Remark 8.21. Similarly let M'_{2i+2} be a trisecant line to $L'_{3i+2} \cup L'_{3i+3} \cup L'_{3i+4}$. Set $B' = (L'_1 \cup \ldots \cup L'_{25+3b}) \cap (M'_1 \cup \ldots \cup M'_{16+2b})$ and observe that $\mathcal{O}_{L'_1 \cup \ldots \cup L'_{25+3b}}(1)(-B') \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus(23+3b)}$. Thus, Lemma 8.9 implies that $L'_1 \cup \ldots \cup L'_{25+3b} \cup M'_1 \cup \ldots \cup M'_{16+2b}$ is nonspecial. We now take a tree of lines $M'_{17+2b} \cup \ldots \cup M'_{35+5b}$ such that M'_i intersects M'_{i-1} at one point, for $17+2b \le i \le 35+5b$. Set

$$Z' = L'_1 \cup \ldots \cup L'_{25+3b} \cup M'_1 \cup \ldots \cup M'_{35+5b}$$

which is a connected stick figure of degree $60+8b$ and genus $8+b$, and it is nonspecial by Lemma 8.9.

We now resume the proof of Claim 8.22.

Observe that, in all cases, Z' is smoothable for being nonspecial, and $p_a(Y \cup Z') = g$. Hence any smoothing $Y \cup Z$ obtained by smoothing Z' so that it still passes through s general points, must have $p_a(Y \cup Z) = g$. That is, again $Y \cap Z = D$, meeting transversally and with two different sets of tangent lines at the s points that are general lines among the ones passing through these points. Since $\mathcal{O}_Y(1)(-D) \cong \mathcal{O}_{\mathbb{P}^1}(61+9b+3\epsilon)$, we have $H^1(\mathcal{O}_Y(1)(-D)) = 0$. Consequently, we see that X' is nonspecial by Lemma 8.9. Finally, the stability of N_Z follows from [BR, Thm. 1]. □

We now conclude the proof of Theorem 3(ii).

Let Y and Z be the curves in Claim 8.22. Let $c = k = 3$ and $s = 22+3b+\epsilon$, where $g = 29+4b+\epsilon$. Note that Y is a general rational curve of degree $83+12b+4\epsilon$ in $\mathbb{P}^4$. Writing $2(83+12b+4\epsilon)-2 = 3\delta+\rho$, with $0 \le \rho \le 2$, it follows by [S, Prop. 2] that

$$N_Y \cong \mathcal{O}_{\mathbb{P}^1}(83+12b+4\epsilon)^{\oplus(3-\rho)} \oplus \mathcal{O}_{\mathbb{P}^1}(84+12b+4\epsilon)^{\oplus\rho}.$$

It is then easily verified that the conditions (i)-(vii) of Lemma 8.5 are satisfied (using (iii-1) and (iv-2)). Moreover, $X' = Y \cup Z$ a nodal nonspecial curve with $\deg X' = 5g-5$ and $p_a(X') = g$. In particular X' is smoothable in $\mathbb{P}^4$. Then a general smoothing $X \subset \mathbb{P}^4$ of X' is such that $\deg X = 5g-5, g(X) = g$ and $N_X^*(3)$ is Ulrich by Lemma 8.5. Also, clearly, X is not subcanonical. □

References

[ACPS] H. Abban, I. Cheltsov, J. Park, C. Shramov. *Double Veronese cones with 28 nodes.* Preprint arXiv:1910.10533. 10

[ALY] A. Atanasov, E. Larson, D. Yang. *Interpolation for normal bundles of general curves.* Mem. Amer. Math. Soc. **257** (2019), no. 1234, 105 pp. 2, 10, 15

[B] A. Beauville. *An introduction to Ulrich bundles.* Eur. J. Math. **4** (2018), no. 1, 26-36. 1

[BE] E. Ballico, Ph. Ellia. *Some more examples of curves in $\mathbb{P}^3$ with stable normal bundle.* J. Reine Angew. Math. **350** (1984), 87-93. 1, 7, 8

[BMPT] V. Benedetti, P. Montero, Y. Prieto Montañez, S. Troncoso. *Projective manifolds whose tangent bundle is Ulrich.* Preprint 2021, arXiv:2108.13944. 1

[BR] E. Ballico, L. Ramella. *On the existence of curves in $\mathbb{P}^n$ with stable normal bundle.* Ann. Polon. Math. 72 (1999), no. 1, 33-42. 2, 18

[C] G. Casnati. *Special Ulrich bundles on non-special surfaces with $p_g = q = 0$.* Internat. J. Math. **28** (2017), no. 8, 1750061, 18 pp. 6

[CH] M. Casanellas, R. Hartshorne. *Stable Ulrich bundles.* With an appendix by F. Geiss, F.-O. Schreyer. Internat. J. Math. **23** (2012), no. 8, 1250083, 50 pp. 6

[CMRPL] L. Costa, R. M. Miró-Roig, J. Pons-Llopis. *Ulrich bundles.* De Gruyter Studies in Mathematics, **77**, De Gruyter 2021. 1

[CLV] I. Coskun, E. Larson, I. Vogt. *Stability of normal bundles of space curves.* Algebra Number Theory **16** (2022), no. 4, 919-953. 15

[CS] I. Cheltsov, C. Shramov. *Cremona groups and the icosahedron.* Monographs and Research Notes in Mathematics. CRC Press, Boca Raton, FL, 2016. xxi+504 pp. 10

[E] L. Ein. *Vanishing theorems for varieties of low codimension.* In: Algebraic geometry (Sundance, UT, 1986), 71-75, Lecture Notes in Math., **1311**, Springer, Berlin, 1988. 5

[EH] G. Ellingsrud, A. Hirschowitz. *Sur le fibré normal des courbes gauches.* C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), no. 7, 245-248. 1, 8

[EL] G. Ellingsrud, D. Laksov. *The normal bundle of elliptic space curves of degree 5*. In: 18th Scandinavian Congress of Mathematicians (Aarhus, 1980), pp. 258-287, Progr. Math., **11**, Birkhäuser, Boston, Mass., 1981. 1, 8

[ES] D. Eisenbud, F.-O. Schreyer. *Resultants and Chow forms via exterior syzygies*. J. Amer. Math. Soc. **16** (2003), no. 3, 537-579. 1

[HH] R. Hartshorne, A. Hirschowitz. *Smoothing algebraic space curves*. In: Algebraic geometry, Sitges (Barcelona), 1983, 98-131, Lecture Notes in Math., **1124**, Springer, Berlin, 1985. 8, 9, 11, 14

[L] A. F. Lopez. *On varieties with Ulrich twisted normal bundles*. Preprint 2022 (arXiv:2205.06602). To appear on *Perspectives on four decades: Algebraic Geometry 1980-2020. In memory of Alberto Collino*. Trends in Mathematics, Birkhäuser. 1

[LR] A. F. Lopez, D. Raychaudhury. *On varieties with Ulrich twisted tangent bundles*. Preprint 2023, arXiv:2301.03104. 1, 4

[P] D. Perrin. *Courbes passant par m points généraux de $\mathbb{P}^3$*. Mém. Soc. Math. France (N.S.) No. **28-29** (1987), 138 pp. 10, 15

[R] Z. Ran. *Curves with semistable normal bundle on Fano hypersurfaces*. Preprint, arXiv:2211.12661. 2

[S] G. Sacchiero. *Fibrati normali di curve razionali dello spazio proiettivo*. Ann. Univ. Ferrara Sez. VII (N.S.) **26** (1980), 33-40 (1981). 10, 14, 18

Vincenzo Antonelli, Dipartimento di Scienze Matematiche, Politecnico di Torino, c.so Duca degli Abruzzi 24, 10129 Torino, Italy. email: vincenzo.antonelli@polito.it

Gianfranco Casnati, Dipartimento di Scienze Matematiche, Politecnico di Torino, c.so Duca degli Abruzzi 24, 10129 Torino, Italy. email: gianfranco.casnati@polito.it

Angelo Felice Lopez, Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy. e-mail lopez@mat.uniroma3.it

Debaditya Raychaudhury, Department of Mathematics, University of Toronto, Bahen Centre, 40 St. George St., Room 6290, Toronto, ON M5S 2E4, Canada. email: debaditya.raychaudhury@utoronto.ca