ON THE CONNECTEDNESS OF SOME DEGENERACY LOCI AND OF ULRICH SUBVARIETIES

VALERIO BUTTINELLI*, ANGELO FELICE LOPEZ AND ROBERTO VACCA**

Abstract. We study connectedness of degeneracy loci $D_{r-k}(\varphi)$ of morphisms $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$, where $\mathcal{E}$ is a rank r globally generated bundle on a smooth n-dimensional variety X and $k \le 3$. For $k \le 2$ we give a characterization of connectedness in terms of vanishing of Chern classes. Moreover we prove that they are connected, for $k \le \min\{2, r-1, n-1\}$, if $\mathcal{E}$ is V-big. In the case of Ulrich bundles more precise results are given, both in general and in the case of surfaces.

1. Introduction

Let $\phi : E \to F$ be a morphism of vector bundles on a smooth complex variety X. A fundamental theorem of Fulton and Lazarsfeld [FuLa] establishes, under the hypothesis that $\mathcal{H}om(E, F)$ is ample, that the degeneracy loci of ϕ are nonempty if they have non-negative expected dimension and are connected if they have positive expected dimension. A few years later, Tu [Tu] and Steffen [Ste] proved a similar result assuming q-ampleness of $\mathcal{H}om(E, F)$. Several results have been proved since then, see for example [D, FU, Lay, LN].

In the absence of some kind of ampleness property, things are more complicated. For example, when $\mathcal{H}om(E, F)$ is just big and globally generated, degeneracy loci can be disconnected [LN] also when they have positive expected dimension (even in the Ulrich case, see Example 10.1 inspired by [LN]).

In this paper we consider, for $1 \le k \le \min\{r, n\}$, injective morphisms

$$\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$$

where $\mathcal{E}$ is a rank r globally generated bundle, and their degeneracy loci

$$D_{r-k}(\varphi) = \{x \in X : \operatorname{rank} \varphi(x) \le r - k\}.$$

which, as is well known, when φ is general, are nonempty if and only if $c_k(\mathcal{E}) \neq 0$.

We first give a small generalization, in this case, of [FuLa, Thm. II(a)] in terms of stable base loci (that measure the positivity of $\mathcal{E}$): if $\mathbf{B}_+(\mathcal{E}) \neq X$, then $D_{r-k}(\varphi)$ is not empty, see Proposition 4.3.

The next natural question to ask is that if one can determine connectedness of $D_{r-k}(\varphi)$ (as a matter of fact, since degeneracy loci are normal, equivalently their irreducibility).

We can give an answer for some k's, as follows.

Theorem 1.

Let X be a smooth irreducible projective variety of dimension n. Let $\mathcal{E}$ be a rank r globally generated bundle on X such that $c_k(\mathcal{E}) \neq 0$ for some integer k and let $s = h^0(\mathcal{E}^*)$, so that $r \ge k + s$. Consider morphisms $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$ such that the degeneracy loci $D_{r-k}(\varphi)$ are reduced of pure codimension k. Also, consider the following conditions:

(i) $c_{k+1}(\mathcal{E}) = 0$.
(ii) *The degeneracy loci $D_{r-k}(\varphi)$ as above have at least $r + 1 - k - s$ connected (or irreducible) components.*
(iii) *The degeneracy loci $D_{r-k}(\varphi)$ as above are disconnected.*

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If $k \in \{1,2\}$ we have: (i) implies (ii), (iii) implies (i), and if $r \ge k+s+1$, then (i), (ii) and (iii) are equivalent. If $k=3$ and $H^1(\mathcal{O}_X)=0$, then (i) implies (ii).

Moreover, under one of the following conditions, for $k \in \{1,2\}$, the degeneracy loci $D_{r-k}(\varphi)$ as above are connected:

(iv) $k \le \min\{r-1, n-1\}$ *and* $\mathbf{B}_+(\mathcal{E}) \ne X$ *(that is $\mathcal{E}$ is V-big).*

(v) φ *is a general morphism and* $D_{r-k}(\varphi)$ *is singular.*

Note that, for any k, the number of connected components is constant, as soon as we require that $D_{r-k}(\varphi)$ is reduced of pure codimension k, see Proposition 6.1.

We point out that the inequality $r \ge k+s+1$, needed to get disconnectedness in (ii), is sharp, see Remark 6.7. Also observe that, in many cases, we can show that the number of connected components is exactly $r+1-k-s$ (see Lemmas 6.5, 6.6 and Remark 6.10).

We are especially interested in the case when $\mathcal{E}$ is an Ulrich vector bundle on a smooth variety $X \subset \mathbb{P}^N$, namely $\mathcal{E}$ satisfies the vanishings $H^i(\mathcal{E}(-p))=0$ for $1 \le p \le \dim X$. Ulrich bundles have emerged as an important class of vector bundles to be studied (see for example [Be, ES, CMRPL]).

In the case $k=2$ when $\mathcal{E}$ is an Ulrich bundle, the degeneracy loci $D_{r-2}(\varphi)$ are known as Ulrich subvarieties (see [LR1] or Definition 5.4) and their existence is equivalent by [LR1, Thm. 1] to the existence of Ulrich bundles, which is still a widely open and considerably studied problem.

We first give the following classification of Ulrich bundles with empty Ulrich subvarieties.

Theorem 2.
Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 2$, degree $d \ge 2$ and let $\mathcal{E}$ be a rank $r \ge 2$ Ulrich bundle on X. The following are equivalent:

(i) All Ulrich subvarieties associated to $\mathcal{E}$ are empty.

(ii) There exists an empty Ulrich subvariety associated to $\mathcal{E}$.

(iii) $c_2(\mathcal{E})=0$.

(iv) $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple over a curve (see [LMS] or Definition 5.3).

Next, as a consequence of Theorem 1, we give a characterization of connectedness of Ulrich subvarieties.

Corollary 1.
Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension n. Let $\mathcal{E}$ be a rank $r \ge 3$ Ulrich bundle on X with $c_2(\mathcal{E}) \ne 0$. Then the following are equivalent:

(i) $c_3(\mathcal{E})=0$.

(ii) *The Ulrich subvarieties associated to $\mathcal{E}$ have at least $r-1$ connected (or irreducible) components.*

(iii) *The Ulrich subvarieties associated to $\mathcal{E}$ are disconnected.*

Moreover, under the following conditions, Ulrich subvarieties Z associated to $\mathcal{E}$ are connected:

(iv) *X is not covered by lines $L \subset X$ such that $\mathcal{E}_{|L}$ is not ample and $n \ge 3$.*

(v) *Z is singular and general (that is $Z = D_{r-2}(\varphi)$, with φ a general morphism).*

In the case of surfaces, Ulrich subvarieties are 0-dimensional smooth subschemes and we can classify completely the connected cases, which correspond to $c_2(\mathcal{E})=1$.

Theorem 3.
Let $S \subset \mathbb{P}^N$ be a smooth irreducible surface of degree $d \ge 2$ and let $\mathcal{E}$ be an Ulrich bundle on S with $c_2(\mathcal{E}) \ne 0$. Then the Ulrich subvarieties associated to $\mathcal{E}$ are connected if and only if $(S, \mathcal{O}_S(1), \mathcal{E})$ is one of the following:

(i) *$(Q, \mathcal{O}_Q(1), \mathcal{S}' \oplus \mathcal{S}'')$, where Q is a smooth quadric in $\mathbb{P}^3$ and $\mathcal{S}', \mathcal{S}''$ are the two spinor bundles on Q.*

(ii) *$(\Gamma, \mathcal{O}_\Gamma(1), \mathcal{O}_\Gamma(T)^{\oplus 2})$, where Γ is a smooth cubic in $\mathbb{P}^3$ and $T \subset \Gamma$ is a twisted cubic.*

(iii) *$(\Sigma, \mathcal{O}_\Sigma(1), \mathcal{O}_\Sigma(C_0+f)^{\oplus 2})$, where Σ is a smooth non-degenerate cubic in $\mathbb{P}^4$.*

We also study some standard cases on 3-folds (see Proposition 9.1) and give some connectedness criteria and several examples with vanishing of Chern classes, see sections 6 and 10.

Finally, in section 11, we prove some results for rank 2 bundles. We have a characterization when $H^1(\mathcal{O}_X)=0$, see Lemma 11.1 and several connectedness or disconnectedness criteria, see Lemma 11.2. These show that, when $H^1(\mathcal{O}_X) \ne 0$, the connectedness scenery appears to be kind of unpredictable.

2. Notation

Unless otherwise specified, we henceforth establish throughout the paper the following.

Notation 2.1.

- X is a smooth irreducible projective variety of dimension $n \geq 1$.
- $\rho(X)$ is the Picard number of X.
- $\nu(\mathcal{L})$ is the numerical dimension of a nef line bundle $\mathcal{L}$ on X.
- $A^k(X)$ is the group rational equivalence classes of $(n-k)$-cycles on X.

  Moreover, when $X \subseteq \mathbb{P}^N$:
- $H \in |\mathcal{O}_X(1)|$ is a very ample divisor.
- $d = H^n$ is the degree of X.
- X is subcanonical if $K_X = -i_X H$ for some $i_X \in \mathbb{Z}$.
- For $1 \leq i \leq n-1$, let $H_i \in |H|$ be general divisors and set $X_n := X$ and $X_i = H_1 \cap \ldots \cap H_{n-i}$.

We work over the complex numbers.

3. Definitions and preliminary results

We will need the following statement on vanishing of cohomology.

Lemma 3.1. *Let A be a divisor on X with $h^0(A) \geq 2$ and let $H = hA$, with $h \geq 1$. Let $\mathcal{F}, \mathcal{G}$ be two vector bundles on X. We have:*

(i) *If $H^0(\mathcal{G}(2A)) = H^1(\mathcal{G}(A)) = 0$, then $H^1(\mathcal{G}) = 0$.*
(ii) *If $H^0(\mathcal{F}(-H)) = H^1(\mathcal{F}(-2H)) = 0$, then $H^1(\mathcal{F}(-jA)) = 0$ for all $j \geq 2h$.*

Proof. Since $h^0(A) \geq 2$, we can choose $B \in |A|$ and then the exact sequence

$$0 \to \mathcal{O}_X \to A \to A_{|B} \to 0$$

implies that $A_{|B}$ is effective. To see (i), observe that the exact sequence

$$0 \to \mathcal{G}(A) \to \mathcal{G}(2A) \to \mathcal{G}(2A)_{|B} \to 0$$

implies that $H^0(\mathcal{G}(2A)_{|B}) = 0$. In particular we have that $\dim B \geq 1$ and, since $H^0(\mathcal{G}(A)_{|B}) \subseteq H^0(\mathcal{G}(2A)_{|B}) = 0$, we deduce that $H^0(\mathcal{G}(A)_{|B}) = 0$. Then, the exact sequence

$$0 \to \mathcal{G} \to \mathcal{G}(A) \to \mathcal{G}(A)_{|B} \to 0$$

implies that $H^1(\mathcal{G}) = 0$. This proves (i). We now show (ii) by induction on j. If $j = 2h$, then $H^1(\mathcal{F}(-jA)) = 0$ by hypothesis. If $j \geq 2h+1$, set $\mathcal{G} = \mathcal{F}(-jA)$. Then, by induction,

$$H^1(\mathcal{G}(A)) = H^1(\mathcal{F}(-(j-1)A)) = 0.$$

Also, since $-j+2 \leq 1-2h \leq -h$ we have that

$$H^0(\mathcal{G}(2A)) = H^0(\mathcal{F}((-j+2)A)) \subseteq H^0(\mathcal{F}(-hA)) = 0.$$

Therefore (i) implies that $H^1(\mathcal{F}(-jA)) = 0$ and this proves (ii). □

Given a vector bundle $\mathcal{E}$ on X, one can measure of the positivity of $\mathcal{E}$ via stable base loci. We recall the relevant definitions (see for example [BKKMSU, Def.'s 2.1 and 2.4]).

Definition 3.2. The *base locus* of $\mathcal{E}$ is $\mathrm{Bs}(\mathcal{E}) = \{x \in X : H^0(\mathcal{E}) \to \mathcal{E}(x) \text{ is not surjective}\}$. The *stable base locus* of $\mathcal{E}$ is $\mathbf{B}(\mathcal{E}) = \bigcap_{m>0} \mathrm{Bs}(S^m\mathcal{E})$. Let A be an ample line bundle on X. The *augmented base locus* of $\mathcal{E}$ is

$$\mathbf{B}_+(\mathcal{E}) = \bigcap_{p/q \in \mathbb{Q}^{>0}} \mathbf{B}((S^q\mathcal{E})(-pA)).$$

The bundle $\mathcal{E}$ is *V-big* if $\mathbf{B}_+(\mathcal{E}) \neq X$.

The definition of $\mathbf{B}_+(\mathcal{E})$ does not depend on the choice of A [BKKMSU, (2.5.1)].

A very useful geometrical vision of vector bundles can be given via degeneracy loci.

Definition 3.3. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a morphism of vector bundles of ranks e, f on X. For any $k \in \mathbb{Z}$ with $0 \le k \le \min\{e, f\}$ we denote the k-th degeneracy locus of φ by $D_k(\varphi)$, that is

$$D_k(\varphi) = \{x \in X : \operatorname{rank} \varphi(x) \le k\}.$$

Remark 3.4. Degeneracy loci have a natural scheme structure, given locally by the vanishings of the $(k+1) \times (k+1)$ minors of the matrix defining $\varphi(x)$. Equivalently [Las, §(2.1)], the ideal sheaf of $D_k(\varphi)$ is the image of the morphism $\Lambda^{k+1}\mathcal{E} \otimes \Lambda^{k+1}\mathcal{F}^* \to \mathcal{O}_X$ induced by $\Lambda^{k+1}\varphi$.

In the following lemma we collect some known results.

Lemma 3.5. *Let $\mathcal{E}, \mathcal{F}$ be vector bundles on X of ranks e, f respectively and such that $\mathcal{E}^* \otimes \mathcal{F}$ is globally generated. Let $\varphi : \mathcal{E} \to \mathcal{F}$ be a general morphism. Let $k \in \mathbb{Z}$ be such that $0 \le k \le \min\{e, f\}$. If $D_k(\varphi) \ne \emptyset$, then $D_k(\varphi)$ is regular in codimension $e + f - 2k$, normal, reduced, Cohen-Macaulay, of pure codimension $(e-k)(f-k)$ and $\operatorname{Sing}(D_k(\varphi)) = D_{k-1}(\varphi)$.*

Proof. It follows from [Ba, Statement (folklore)(i), §4.1] that $D_k(\varphi)$ is of pure codimension $(e-k)(f-k)$ and $\operatorname{Sing}(D_k(\varphi)) = D_{k-1}(\varphi)$. Then, $D_k(\varphi)$ is Cohen-Macaulay by [ACGH, Ch. II, Prop. (4.1)]. If $D_k(\varphi)$ is smooth we are done. Otherwise, [Ba, Statement (folklore)(i), §4.1] implies that $\operatorname{Sing}(D_{k-1}(\varphi)) \ne \emptyset$ is of pure codimension $(e+1-k)(f+1-k)$. In particular $n - (e-k)(f-k) \ge e + f - 2k + 1 \ge 2$. Now $\operatorname{codim}_{D_k(\varphi)} D_{k-1}(\varphi) = e + f - 2k + 1 \ge 2$ so that $D_k(\varphi)$ is regular in codimension $e + f - 2k$ and is R_1. Moreover, for any $x \in D_k(\varphi)$ and any prime ideal $\mathfrak{p}$ of $A := \mathcal{O}_{D_k(\varphi),x}$ we have that $A_{\mathfrak{p}}$ is Cohen-Macaulay by [H1, Thm. II.8.21A(b)]. Hence $\operatorname{depth} A_{\mathfrak{p}} = \dim A_{\mathfrak{p}} = n - (e-k)(f-k) \ge 2$, so that A is S_2. Then A is normal by [H1, Thm. II.8.22A] and reduced by [ST, Lemma 10.153.3, Tag 031R], whence so is $D_k(\varphi)$. □

4. Generalities on globally generated bundles

We collect here some simple but useful results about globally generated bundles.

Lemma 4.1. *Let $\mathcal{E}$ be a rank $r \ge 1$ globally generated bundle on X. For any $1 \le i \le n$, let H be a very ample divisor on X. We have:*

(i) *$c_1(\mathcal{E})^i = 0$ if and only if $c_1(\mathcal{E})^i H^{n-i} = 0$.*
(ii) *$c_i(\mathcal{E}) = 0$ if and only if $c_i(\mathcal{E}) H^{n-i} = 0$.*

Moreover, let $k \in \mathbb{Z}$ be such that $1 \le k \le \min\{r, n\}$. Let $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$ be a general morphism. Then the following are equivalent:

(iii) *$D_{r-k}(\varphi) = \emptyset$.*
(iv) *$c_i(\mathcal{E}) = 0$ for $i \ge k$.*
(v) *$c_k(\mathcal{E}) = 0$.*

Also, if any of (iii)-(v) does not hold, then $D_{r-k}(\varphi)$ has pure codimension k and $c_k(\mathcal{E}) = [D_{r-k}(\varphi)] \in A^k(X)$.

Proof. To see (i), one implication being obvious, assume that $c_1(\mathcal{E})^i H^{n-i} = 0$. Since $\mathcal{E}$ is globally generated, it follows that $\det \mathcal{E}$ is globally generated and $(\det \mathcal{E})^i H^{n-i} = 0$ implies that $c_1(\mathcal{E})^i = 0$ by [FuLe2, Cor. 3.15] (see also [FuLe1, Prop. 3.7]). This proves (i). Suppose now that $D_{r-k}(\varphi) \ne \emptyset$. Then $D_{r-k}(\varphi)$ has pure dimension $n - k \ge 0$ by Lemma 3.5, so that $c_k(\mathcal{E}) = [D_{r-k}(\varphi)] \in A^k(X)$. This proves the last assertion of the lemma, once proved the equivalence (iii)-(v), that we now show. If (iii) holds, we have that φ has constant rank $r + 1 - k$ and we get an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is also a vector bundle, of rank $k - 1$. But then, for any $i \ge k$ we have

$$c_i(\mathcal{E}) = \sum_{j=0}^{i} c_j(\mathcal{O}_X^{\oplus(r+1-k)}) c_{i-j}(\mathcal{F}) = c_i(\mathcal{F}) = 0.$$

Hence (iii) implies (iv). Clearly (iv) implies (v). Now assume (v). If $D_{r-k}(\varphi) \ne \emptyset$ we get a contradiction both if $n - k = 0$, since $0 = c_k(\mathcal{E}) = [D_{r-k}(\varphi)]$, while $D_{r-k}(\varphi)$ is 0-dimensional and nonempty and if $n - k > 0$, since we would have that $0 < H^k D_{r-k}(\varphi) = H^k c_k(\mathcal{E}) = 0$. This proves that (v) implies (iii), hence the equivalence (iii)-(v) and also proves the last assertion of the lemma. It remains to prove (ii). One implication being obvious, we assume that $c_i(\mathcal{E})H^{n-i} = 0$. If $i = n$ we are done, so assume that $i \le n-1$. Setting $k = i$, if $c_i(\mathcal{E}) \ne 0$, we get that $D_{r-i}(\varphi)$ is nonempty of pure codimension i and $c_i(\mathcal{E}) = [D_{r-i}(\varphi)]$, giving the contradiction $0 < H^{n-i}D_{r-i}(\varphi) = H^{n-i}c_i(\mathcal{E}) = 0$. This proves (ii) and concludes the proof of the lemma. □

In the sequel, we will be interested in the vanishing of some Chern classes, which we now study.

Lemma 4.2. *Let $\mathcal{E}$ be a rank r globally generated bundle on $X \subset \mathbb{P}^N$. We have:*

(i) *If $c_1(\mathcal{E})^t = 0$ for some $t \ge 1$, then $c_i(\mathcal{E}) = 0$ for all $i \ge t$.*
(ii) *If $t \ge 1$ and $c_1(\mathcal{E})^t \ne 0$ (in particular if $c_t(\mathcal{E}) \ne 0$), then $H^j(-\det \mathcal{E}) = 0$ for $0 \le j \le t-1$.*
(iii) *Let $s = h^0(\mathcal{E}^*)$. If $s \ge 1$, then $\mathcal{E} \cong \mathcal{O}_X^{\oplus s} \oplus \mathcal{E}_1$ with $\mathcal{E}_1$ globally generated of rank $r-s$ and $h^0(\mathcal{E}_1^*) = 0$. Moreover, if $c_k(\mathcal{E}) \ne 0$, then $r \ge k+s$.*

Proof. Assume that $c_1(\mathcal{E})^t = 0$. The conclusion (i) being obvious if $n \le t-1$, suppose that $n \ge t$. Then $c_1(\mathcal{E}_{|X_t})^t = c_1(\mathcal{E})^t H^{n-t} = 0$, hence $c_t(\mathcal{E})H^{n-t} = c_t(\mathcal{E}_{|X_t}) = 0$ by [DPS, Cor. 2.7]. Therefore $c_t(\mathcal{E}) = 0$ by Lemma 4.1(ii) and then (i) holds by the same lemma. To see (ii), first observe that if $c_t(\mathcal{E}) \ne 0$, then $c_1(\mathcal{E})^t \ne 0$ by (i). Hence, to prove (ii), we can assume that $c_1(\mathcal{E})^t \ne 0$, and we have, in particular, that $n \ge t$. Set $\det \mathcal{E} = \mathcal{O}_X(D)$. We prove that $H^j(\mathcal{O}_X(-D)) = 0$ for $0 \le j \le t-1$ by induction on n. If $n = t$, we know that $D^t > 0$, hence D is big and nef and therefore $H^j(\mathcal{O}_X(-D)) = 0$ by Kawamata-Viehweg's vanishing theorem. If $n \ge t+1$, note that $c_1(\mathcal{E}_{|X_{n-1}})^t = c_1(\mathcal{E})^t H \ne 0$, for otherwise also $c_1(\mathcal{E})^t H^{n-t} = 0$, implying, by Lemma 4.1(ii), the contradiction $c_1(\mathcal{E})^t = 0$. Then, in the exact sequence

$$0 \to \mathcal{O}_X(-D-H) \to \mathcal{O}_X(-D) \to \mathcal{O}_{X_{n-1}}(-D) \to 0$$

we have that $H^j(\mathcal{O}_{X_{n-1}}(-D)) = H^j(-\det(\mathcal{E}_{|X_{n-1}})) = 0$ by the inductive hypothesis and also $H^j(\mathcal{O}_X(-D-H)) = 0$ by Kodaira vanishing. Hence we find that $H^j(\mathcal{O}_X(-D)) = 0$, proving (ii). Finally, the first part of (iii) is well-known (see for example [SU, Proof of Lemma 3] or use [O2, Lemma 3.9] and induction). Now, when $c_k(\mathcal{E}) \ne 0$ we have that $c_k(\mathcal{E}_1) = c_k(\mathcal{E}) \ne 0$ and therefore $r-s \ge k$. □

We observe that V-bigness implies non-vanishing of Chern classes, hence also non-emptiness and connectedness (via Theorem 1).

Proposition 4.3. *Let k be an integer such that $1 \le k \le \min\{r,n\}$. Let $\mathcal{E}$ be a rank r vector bundle on X with $\mathbf{B}_+(\mathcal{E}) \ne X$. We have:*

(i) *If $\mathcal{E}$ is nef, then $c_k(\mathcal{E}) \ne 0$.*
(ii) *If $\mathcal{E}$ is globally generated, then $H^0(\mathcal{E}^*) = 0$.*
(iii) *If $\mathcal{E}$ is globally generated, then $D_{r-k}(\varphi) \ne \emptyset$ for any injective morphism $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$.*

Proof. We will use $\mathbb{Q}$-twisted bundles and their Chern classes, see [Laz2, §6.2, 8.1, 8.2]. We first observe that if $\mathcal{F}\langle\delta\rangle$ is a $\mathbb{Q}$-twisted nef bundle on X, then $c_k(\mathcal{F}\langle\delta\rangle)$ is nef. Indeed, this follows from [DPS, Proof of Prop. 2.1 and Cor. 2.2], since the proof works for $\mathbb{R}$-twisted nef vector bundles. Alternatively, for any k-dimensional irreducible subvariety $Y \subset X$, the restriction $(\mathcal{F}\langle\delta\rangle)_{|Y}$ is still nef by [Laz2, Thm. 6.2.12(v)]. Then, it is enough to apply [Laz2, Thm. 8.2.1] to obtain that $c_k(\mathcal{F}\langle\delta\rangle)\cdot Y = c_k((\mathcal{F}\langle\delta\rangle)_{|Y}) \ge 0$, as desired. We now use Seshadri constants $\varepsilon(\mathcal{E};x)$, in particular [FM, Rmk. 3.10(c)]. To see (i), let $x \notin \mathbf{B}_+(\mathcal{E})$ and let $\mu : \widetilde{X} \to X$ be the blowing up of X at x, with exceptional divisor E. It follows from [FM, Prop. 6.9 and Rmk. 3.10(c)] that $\varepsilon(\mathcal{E};x) > 0$, hence it is easy to see that we can find $t \in \mathbb{Q}_{>0}$ such that the $\mathbb{Q}$-twisted vector bundle $(\mu^*\mathcal{E})\langle -tE\rangle$ is nef. If follows from the observation above, that $c_k((\mu^*\mathcal{E})\langle -tE\rangle)$ is nef. Let Z be any irreducible subvariety with $x \in Z \subseteq X, \dim Z = k$ and consider its strict transform $\widetilde{Z}$ on $\widetilde{X}$. Then, as in [Lo, Proof of Thm. 7.2], we have

$$0 \le c_k((\mu^*\mathcal{E})\langle -tE\rangle) \cdot \widetilde{Z} = c_k(\mathcal{E}) \cdot Z - t^k \binom{r}{k} \operatorname{mult}_x(Z)$$

so that $c_k(\mathcal{E}) \ne 0$ and (i) is proved. To see (ii), assume that $s = h^0(\mathcal{E}^*) \ge 1$. By Lemma 4.2(iii) we have $\mathcal{E} = \mathcal{O}_X^{\oplus s} \oplus \mathcal{E}_1$, with $\mathcal{E}_1$ a globally generated bundle. Since $\mathcal{E}$ is nef and $\mathbf{B}_+(\mathcal{E}) \ne X$, it follows by

[FM, Prop. 6.9] that there is $x \in X$ such that $\varepsilon(\mathcal{E}; x) > 0$. However, as $\varepsilon(\mathcal{E}_1; x) \geq 0$ by the nefness of $\mathcal{E}_1$ [FM, Rmk. 3.10(a)], we get by [FM, Lemma 3.31] that

$$0 < \varepsilon(\mathcal{E}; x) = \min\{\varepsilon(\mathcal{O}_X^{\oplus s}; x), \varepsilon(\mathcal{E}_1; x)\} = \varepsilon(\mathcal{O}_X^{\oplus s}; x) = 0$$

a contradiction. This proves (ii). As for (iii), assume that $D_{r-k}(\varphi) = \emptyset$. If $k = 1$ then φ is an isomorphism and we get that $c_1(\mathcal{E}) = 0$, contradicting (i). If $k \geq 2$, we have an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a rank $k-1$ bundle on X, hence $c_k(\mathcal{E}) = c_k(\mathcal{F}) = 0$, contradicting (i). This proves (iii). □

We add a calculation of the n-th Segre class. This will be useful to detect bigness in some examples.

Lemma 4.4. *Let $n \geq 2$ and let $\mathcal{E}$ be a globally generated bundle on X such that $c_2(\mathcal{E})^2 = 0$ and $c_3(\mathcal{E}) = 0$. Then*

$$s_n(\mathcal{E}^*) = c_1(\mathcal{E})^n - (n-1)c_1(\mathcal{E})^{n-2}c_2(\mathcal{E}).$$

In particular $\mathcal{E}$ is big if and only if $c_1(\mathcal{E})^n > (n-1)c_1(\mathcal{E})^{n-2}c_2(\mathcal{E})$.

Proof. For $n \geq 2$, consider the $n \times n$ determinant

$$P_n(x_1, x_2) = \begin{vmatrix} x_1 & x_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & x_1 & x_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & x_1 & x_2 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & x_1 \end{vmatrix}.$$

It is easily seen that $P_2(x_1, x_2) = x_1^2 - x_2$, $P_3(x_1, x_2) = x_1^3 - 2x_1x_2$ and that, for $n \geq 4$,

$$P_n(x_1, x_2) = x_1 P_{n-1}(x_1, x_2) - x_2 P_{n-2}(x_1, x_2). \tag{4.1}$$

Setting $c_i = c_i(\mathcal{E})$ and using (4.1), it follows by induction on $n \geq 2$ that

$$P_n(c_1, c_2) = c_1^n - (n-1)c_1^{n-2}c_2.$$

On the other hand, we have that $c_i(\mathcal{E}) = 0$ for $i \geq 3$ by Lemma 4.1, hence $P_n(c_1, c_2)$ is just the Schur polynomial of $\mathcal{E}$ associated to the partition $1^n = (1, \dots, 1)$ and therefore (see for example [Laz2, Exa. 8.3.5]), we get that

$$s_n(\mathcal{E}^*) = s_{1^n}(\mathcal{E}) = P_n(c_1, c_2) = c_1^n - (n-1)c_1^{n-2}c_2.$$

As is well-known (see for example [LM, Rmk. 2.2]), $\mathcal{E}$ is big if and only if $s_n(\mathcal{E}^*) > 0$, thus if and only if $c_1(\mathcal{E})^n > (n-1)c_1(\mathcal{E})^{n-2}c_2(\mathcal{E})$. □

The following lemma, relating some degeneracy loci (see also [CFK, Rmk. 3.4]), will be crucial in the proof of Theorem 1.

Lemma 4.5. *Let $n \geq 2$ and let $\mathcal{E}$ be a rank r globally generated bundle on X such that $c_2(\mathcal{E}) \neq 0$. Let $V \subset H^0(\mathcal{E})$ be a general subspace with $\dim V = r-1$, let $\varphi : V \otimes \mathcal{O}_X \to \mathcal{E}$, so that $Z = D_{r-2}(\varphi) \neq \emptyset$. Then there is a normal irreducible Cartier divisor $Y \in |\det(\mathcal{E})|$, smooth if $n \leq 3$, such that $Z \subset Y$. Moreover, if Y is smooth, we have an exact sequence*

$$0 \to \mathcal{E}^* \to \mathcal{O}_X^{\oplus r} \to \mathcal{O}_Y(Z) \to 0 \tag{4.2}$$

and $\mathcal{O}_Y(Z)$ is globally generated by at most r sections. Also, using, on the right-hand side, the intersection product on Y and a very ample divisor H, we have, if $n \geq 3$, that

$$c_3(\mathcal{E})H^{n-3} = Z^2 H_{|Y}^{n-3}. \tag{4.3}$$

Proof. Note that $Z \neq \emptyset$ by Lemma 4.1, so that Z is reduced of pure codimension 2 by Lemma 3.5. Moreover, if Z is singular, then $\mathrm{Sing}(Z) = D_{r-3}(\varphi)$. Also, the Eagon-Northcott complex gives is an exact sequence

$$0 \to V \otimes \mathcal{O}_X \to \mathcal{E} \to \mathcal{J}_{Z/X}(D) \to 0. \tag{4.4}$$

Choose a general subspace $V' \subset H^0(\mathcal{E})$ such that $\dim V' = r$ and $V \subset V'$. Thus, we get a general morphism $\varphi' : V' \otimes \mathcal{O}_X \to \mathcal{E}$ and setting $Y = D_{r-1}(\varphi')$ we see that $Z \subset Y$: If $x \in Z$, then $\operatorname{rank} \varphi(x) \leq$

$r-2$, hence $\operatorname{rank}\varphi'(x) \le r-1$, so that $x \in Y$. In particular Y is nonempty and Lemma 3.5 gives that Y is smooth if $n \le 3$. Note that $c_1(\mathcal{E})^2 \ne 0$, for otherwise $c_2(\mathcal{E}) = 0$ by Lemma 4.2(i). Therefore the morphism $\varphi_{\det \mathcal{E}} : X \to \mathbb{P}H^0(\det \mathcal{E})$ is not composed with a pencil and $Y \in |\det(\mathcal{E})|$ is connected by Bertini's theorem. Since Y is normal by Lemma 3.5, it is irreducible. Now assume that Y is smooth, so that $D_{r-2}(\varphi') = \emptyset$ by Lemma 3.5. Moreover also Z is smooth, because, as above, we have that $D_{r-3}(\varphi) \subset D_{r-2}(\varphi')$. Using (4.4), we have a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V \otimes \mathcal{O}_X \arrow[r, "\varphi"] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{J}_{Z/X}(D) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V' \otimes \mathcal{O}_X \arrow[r, "\varphi'"] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{L} \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_X \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

where $\mathcal{L}$ is a sheaf supported on Y. The diagram shows that $\mathcal{L}$ is a line bundle on Y, since $\operatorname{rank}\varphi'(y) = r-1$ for every $y \in Y$. Also, we get an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{J}_{Z/X}(D) \to \mathcal{L} \to 0$$

and the map $\mathcal{O}_X \to \mathcal{J}_{Z/X}(D)$ is given by the section defining Y. Therefore $\mathcal{L} \cong \mathcal{J}_{Z/Y}(D)$. We will now use the well-known fact that the proof of [H1, Lemma III.7.4] works also for the sheaf version with $\mathcal{H}om$ and $\mathcal{E}xt$. Using it, we get

$$\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \cong \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{L}, \omega_X)(-K_X) \cong \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{L}, \omega_Y)(-K_X) \cong \mathcal{O}_Y(K_Y + Z - D)(-K_X) \cong \mathcal{O}_Y(Z).$$

Hence, dualizing the exact sequence, obtained in the diagram above,

$$0 \to V' \otimes \mathcal{O}_X \to \mathcal{E} \to \mathcal{L} \to 0$$

we get the exact sequence

$$0 \to \mathcal{E}^* \to \mathcal{O}_X^{\oplus r} \to \mathcal{O}_Y(Z) \to 0$$

showing (4.2) and that $\mathcal{O}_Y(Z)$ is globally generated by at most r sections. Finally, to see (4.3), if $n = 3$, we set $X' = X, \mathcal{E}' = \mathcal{E}$ and $Z' = Z$. If $n \ge 4$, cutting down with $n-3$ general $H_1, \dots, H_{n-3} \in |H|$, we get a smooth 3-fold $X' = X_3$ and a globally generated bundle $\mathcal{E}' = \mathcal{E}_{|X_3}$. Setting $Z' = Z \cap H_1 \cap \dots \cap H_{n-3}$, we have that

$$Z' = Z \cap X' = D_{r-2}(\varphi) \cap X' = D_{r-2}(\varphi_{|X'})$$

so that, in particular, $[Z'] = c_2(\mathcal{E}')$. Moreover, setting $Y' = Y \cap H_1 \cap \dots \cap H_{n-3}$, we see that $Y' \in |\det \mathcal{E}'|$ is a smooth irreducible surface containing Z' and (4.2) gives an exact sequence

$$0 \to (\mathcal{E}')^* \to \mathcal{O}_{X'}^{\oplus r} \to \mathcal{O}_{Y'}(Z') \to 0. \tag{4.5}$$

We compute the Euler characteristics in (4.5). To this end, we set $c_i = c_i(X')$ and $d_i = c_i(\mathcal{E}')$, for $1 \le i \le 3$. Now, Riemann-Roch on X' gives

$$\chi((\mathcal{E}')^*) = r\chi(\mathcal{O}_{X'}) - \frac{1}{12}d_1(c_1^2 + c_2) + \frac{1}{4}c_1(d_1^2 - 2d_2) - \frac{1}{6}(d_1^3 - 3d_1d_2 + 3d_3). \tag{4.6}$$

The exact sequence

$$0 \to \mathcal{O}_{X'}(-Y') \to \mathcal{O}_{X'} \to \mathcal{O}_{Y'} \to 0 \tag{4.7}$$

gives, using Riemann-Roch on X',

$$\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_{X'}) - \chi(\mathcal{O}_{X'}(-Y')) = \frac{1}{12}d_1(d_1 - c_1)(2d_1 - c_1) + \frac{1}{12}d_1c_2. \tag{4.8}$$

Moreover, by Riemann-Roch on Y', we get, using adjunction, (4.8) and $[Z'] = c_2(\mathcal{E}')$,

$$\chi(\mathcal{O}_{Y'}(Z')) = \chi(\mathcal{O}_{Y'}) + \frac{1}{2}Z'(Z' - K_{Y'}) = \chi(\mathcal{O}_{Y'}) + \frac{1}{2}d_2(c_1 - d_1) + \frac{1}{2}(Z')^2 = \tag{4.9}$$
$$= \frac{1}{12}d_1(d_1 - c_1)(2d_1 - c_1) + \frac{1}{12}d_1c_2 + \frac{1}{2}d_2(c_1 - d_1) + \frac{1}{2}(Z')^2.$$

Therefore (4.5), together with (4.6) and (4.9), gives

$$r\chi(\mathcal{O}_{X'}) = \chi((\mathcal{E}')^*) + \chi(\mathcal{O}_{Y'}(Z')) = r\chi(\mathcal{O}_{X'}) - \frac{1}{12}d_1(c_1^2 + c_2) + \frac{1}{4}c_1(d_1^2 - 2d_2) - \frac{1}{6}(d_1^3 - 3d_1d_2 + 3d_3) +$$
$$+ \frac{1}{12}d_1(d_1 - c_1)(2d_1 - c_1) + \frac{1}{12}d_1c_2 + \frac{1}{2}d_2(c_1 - d_1) + \frac{1}{2}(Z')^2 = r\chi(\mathcal{O}_{X'}) - \frac{1}{2}d_3 + \frac{1}{2}(Z')^2.$$

Hence $c_3(\mathcal{E})H^{n-3} = c_3(\mathcal{E}') = d_3 = (Z')^2 = Z^2H_{|Y}^{n-3}$ and (4.3) is proved. □

We now give a fact that will be useful to study Ulrich subvarieties in the case of a linear Ulrich triple.

Lemma 4.6. *Let X, B be two projective varieties with X smooth and let $\pi : X \to B$ be a flat morphism with $\pi_*\mathcal{O}_X \cong \mathcal{O}_B$. Let $\mathcal{E}_B$ be a rank r bundle on B such that $\mathcal{E} = \pi^*\mathcal{E}_B$ is globally generated. Let $k \in \mathbb{Z}$ be such that $1 \le k \le r$, let $V \subseteq H^0(\mathcal{E})$ be a general subspace of dimension $r + 1 - k$ and, if $\varphi : V \otimes \mathcal{O}_X \to \mathcal{E}$, assume that $Z := D_{r-k}(\varphi)$ is nonempty. Then there exists $V_B \subset H^0(\mathcal{E}_B)$ a general subspace such that $V = \pi^*V_B$ and, if $\varphi_B : V_B \otimes \mathcal{O}_B \to \mathcal{E}_B$ is the associated morphism and $Z_B = D_{r-k}(\varphi_B)$, we have that $\mathcal{O}_Z \cong \pi^*\mathcal{O}_{Z_B}$, Z is the scheme-theoretic inverse image of Z_B under π and Z_B has pure codimension k in B.*

Proof. π is surjective being both open and closed. Note that B is smooth and $\mathcal{E}_B$ is globally generated since X and $\mathcal{E} = \pi^*\mathcal{E}_B$ respectively are (see for example [Li, Rmk. 4.3.25 and Exc. 5.1.29(b)]). Then, by projection formula, we obtain

$$H^0(\mathcal{E}) \cong H^0(\mathcal{E}_B \otimes \pi_*\mathcal{O}_X) \cong H^0(\mathcal{E}_B).$$

Moreover, as the pull-back of sections $\pi^* \colon H^0(\mathcal{E}_B) \to H^0(\mathcal{E})$ is injective by [G1, Cor. 2.2.8], the above isomorphism is induced by π^*. We set $V_B \subseteq H^0(\mathcal{E}_B)$ to be the subspace such that $V = \pi^*V_B$. Then $V_B \subseteq H^0(\mathcal{E}_B)$ is general and we get that $Z_B \ne \emptyset$: In fact, if not, we would have that $c_k(\mathcal{E}_B) = 0$ by Lemma 4.1, and therefore also $c_k(\mathcal{E}) = \pi^*c_k(\mathcal{E}_B) = 0$, giving, by Lemma 4.1 again, the contradiction $Z = \emptyset$. Therefore Z_B has pure codimension k in B by Lemma 3.5. It follows that we have the Eagon-Northcott resolution

$$0 \to S^{k-1}V_B \otimes \mathcal{O}_B \to \cdots \to V_B \otimes \Lambda^{k-2}\mathcal{E}_B \to \Lambda^{k-1}\mathcal{E}_B \to \mathcal{J}_{Z_B/B}(\det \mathcal{E}_B) \to 0$$

whose pull-back via π is, since $V \cong V_B$ via pull-back of sections, the exact sequence

$$0 \to S^{k-1}V \otimes \mathcal{O}_X \to \cdots \to V \otimes \Lambda^{k-2}\mathcal{E} \to \Lambda^{k-1}\mathcal{E} \to \pi^*(\mathcal{J}_{Z_B/B}(\det \mathcal{E}_B)) \to 0.$$

On the other hand, we have the analogous resolution

$$0 \to S^{k-1}V \otimes \mathcal{O}_X \to \cdots \to V \otimes \Lambda^{k-2}\mathcal{E} \to \Lambda^{k-1}\mathcal{E} \to \mathcal{J}_{Z/X}(D) \to 0$$

where $D = \det \mathcal{E} = \pi^*(\det \mathcal{E}_B)$. We get that $\mathcal{J}_{Z/X}(D) \cong \pi^*(\mathcal{J}_{Z_B/B}(\det \mathcal{E}_B))$ and therefore

$$\mathcal{J}_{Z/X} \cong \pi^*\mathcal{J}_{Z_B/B}. \tag{4.10}$$

Now, pulling back the exact sequence

$$0 \to \mathcal{J}_{Z_B/B} \to \mathcal{O}_B \to \mathcal{O}_{Z_B} \to 0$$

we get an exact sequence

$$0 \to \pi^*\mathcal{J}_{Z_B/B} \to \mathcal{O}_X \to \pi^*\mathcal{O}_{Z_B} \to 0$$

and (4.10) shows that $\mathcal{O}_Z \cong \pi^*\mathcal{O}_{Z_B}$. Since π is flat, we also have (see for example [ST, Lemma 26.4.7, Tag 01HQ]) that Z is the scheme-theoretic inverse image of Z_B. □

5. Generalities on Ulrich vector bundles

We will often use the following well-known properties of Ulrich bundles

Lemma 5.1. *Let $\mathcal{E}$ be a rank r Ulrich bundle on $X \subset \mathbb{P}^N$. We have:*

(i) *$\mathcal{E}$ is globally generated.*
(ii) *$\mathcal{E}_{|X_{n-1}}$ is Ulrich on a smooth hyperplane section X_{n-1} of X.*
(iii) *$\det \mathcal{E}$ is globally generated and it is not trivial, unless $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.*
(iv) *$H^0(\mathcal{E}^*) = 0$, unless $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}^{\oplus r})$.*
(v) *$\mathcal{O}_X(l)$ is Ulrich if and only if $(X, \mathcal{O}_X(1), l) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), 0)$.*
(vi) *$\mathcal{E}$ is aCM.*
(vii) *$h^0(\mathcal{E}) = rd$.*
(viii) *If $(X, \mathcal{O}_X(1)) = (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, then $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$.*
(ix) *$\mathcal{E}^*(K_X + (n+1)H)$ is Ulrich.*
(x) *$c_1(\mathcal{E})H^{n-1} = \frac{r}{2}[K_X + (n+1)H]H^{n-1}$.*

Proof. For (i)-(vi) and (x), see for example [LR1, Lemma 3.1]. For (vii) see [ES, Prop. 2.1] (or [Be, Thm. 2.3]) and for (viii) see [Be, (3.1)]. (ix) follows by definition of Ulrich and Serre's duality. □

We also recall the following examples of Ulrich bundles.

Definition 5.2. For $n \ge 2$, we let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric. We let S (n odd), and S', S'' (n even), be the vector bundles on Q_n, as defined in [O1, Def. 1.3]. The *spinor bundles* on Q_n are $\mathcal{S} = \mathcal{S}_n = S(1)$ if n is odd and $\mathcal{S}' = \mathcal{S}'_n = S'(1)$, $\mathcal{S}'' = \mathcal{S}''_n = S''(1)$, if n is even.

Definition 5.3. Let $\mathcal{E}$ be a vector bundle on $X \subset \mathbb{P}^N$. We say that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a *linear Ulrich triple* if there are a smooth irreducible variety B of dimension $b \ge 1$, a very ample vector bundle $\mathcal{F}$ and a vector bundle $\mathcal{G}$ on B such that $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}(\mathcal{F}), \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1), \pi^*(\mathcal{G}(\det \mathcal{F})))$, where $\pi : X \cong \mathbb{P}(\mathcal{F}) \to B$ is the bundle map and $H^j(\mathcal{G} \otimes S^k\mathcal{F}^*) = 0$ for all $j \ge 0, 0 \le k \le b-1$.

When $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple, then $\mathcal{E}$ is an Ulrich bundle on X by [Lo, Lemma 4.1]. Next, we recall some definitions and facts in [LR1].

Definition 5.4. Let $n \ge 2$ and $d \ge 2$. Let $\mathcal{E}$ be a rank $r \ge 2$ Ulrich bundle on $X \subset \mathbb{P}^N$. Let $V \subset H^0(\mathcal{E})$ be a subspace of dimension $r-1$ such that, if $\varphi : V \otimes \mathcal{O}_X \to \mathcal{E}$ is the associated morphism and $Z = D_{r-2}(\varphi)$, then Z satisfies the following conditions (in particular these hold, by Lemma 3.5, if V is a general subspace of $H^0(\mathcal{E})$):

(a) Z is either empty or of pure codimension 2,
(b) if $Z \ne \emptyset$ and either $r = 2$ or $n \le 5$, then Z is smooth (possibly disconnected),
(c) if $Z \ne \emptyset$ and $n \ge 6$, then Z is either smooth or is normal, Cohen-Macaulay, reduced and with $\dim \operatorname{Sing}(Z) = n-6$.

We call Z *an Ulrich subvariety associated to* $\mathcal{E}$. We say that Z is *a general Ulrich subvariety associated to* $\mathcal{E}$ if V is a general subspace of $H^0(\mathcal{E})$.

Remark 5.5. Let $Z \subset X$ be any subvariety satisfying (a)-(c) above and (i)-(vi) of [LR1, Thm. 1]. It follows from [LR1, Thm. 1] that there is an Ulrich bundle $\mathcal{E}$ such that Z is associated to $\mathcal{E}$.

We now prove Theorem 2.

Proof of Theorem 2. Ulrich subvarieties exist by [LR1, Thm. 1], therefore (i) implies (ii). If (ii) holds, there is an empty Ulrich subvariety associated to $\mathcal{E}$, hence $c_2(\mathcal{E}) = 0$ by Lemma 4.1, that is (iii). Next, assume (iii). Let $S = X_2, \mathcal{E}' = \mathcal{E}_{|S}, D' \in |\det \mathcal{E}'|$. We have that $\mathcal{E}'$ is a rank r Ulrich bundle on S by Lemma 5.1(ii) and $\mathcal{E}'$ is globally generated by Lemma 5.1(i). Also, $c_2(\mathcal{E}') = c_2(\mathcal{E})_{|S} = c_2(\mathcal{E}) \cdot H^{n-2} = 0$. Choosing $r-1$ general sections in $H^0(\mathcal{E}')$ we get a general morphism $\varphi : \mathcal{O}_S^{\oplus(r-1)} \to \mathcal{E}'$ and $D_{r-2}(\varphi) = \emptyset$ by Lemma 4.1 with $k = 2$. It follows that we have an exact sequence

$$0 \to \mathcal{O}_S^{\oplus(r-1)} \to \mathcal{E}' \to \mathcal{O}_S(D') \to 0. \tag{5.1}$$

Now

$$\operatorname{Ext}^1(\mathcal{O}_S(D')), \mathcal{O}_S^{\oplus(r-1)}) \cong H^1(\mathcal{O}_S(-D'))^{\oplus(r-1)}. \tag{5.2}$$

By Lemma 5.1(iii) we have that $\mathcal{O}_S(D')$ is globally generated, hence nef. If $(D')^2 > 0$, then D' is big and Kawamata-Viehweg's vanishing theorem shows that $H^1(\mathcal{O}_S(-D'))) = 0$. Hence (5.2) implies that (5.1) splits, $\mathcal{E}' \cong \mathcal{O}_S^{\oplus(r-1)} \oplus \mathcal{O}_S(D')$ and therefore $\mathcal{O}_S$ is Ulrich. It follows from Lemma 5.1(v) that $(S, \mathcal{O}_S(1)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))$, hence $d = 1$, a contradiction. Therefore $(D')^2 = 0$. Then $c_1(\mathcal{E})^2 \cdot H^{n-2} = (D')^2 = 0$, so that $c_1(\mathcal{E})^2 = 0$ by Lemma 4.1(i). Note that $c_1(\mathcal{E}) \neq 0$, for otherwise $d = 1$ by [Lo, Lemma 2.1]. Hence $\nu(\det(\mathcal{E})) = 1$ and, if $\Phi : X \to \mathbb{G}(r-1, \mathbb{P}H^0(\mathcal{E}))$ and $F_x = \Phi^{-1}(\Phi(x))$, then $\dim F_x = n-1$ for every $x \in X$ by [LS, Thm. 2]. Therefore $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (iv) by [LMS, Lemmas 2.10 and 2.12].

Finally, if $(X, \mathcal{O}_X(1), \mathcal{E})$ is as in (iv), then $c_2(\mathcal{E}) = 0$ and therefore, if Z is any Ulrich subvariety associated to $\mathcal{E}$, we have that $Z = D_{r-2}(\varphi)$ for some morphism $\varphi : V \otimes \mathcal{O}_X \to \mathcal{E}$ by Remark 5.4. Hence $Z = \emptyset$ by Lemma 4.1. Thus (iv) implies (i) and we are done. □

We observe the following simple consequence of Theorem 2.

Remark 5.6. Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \geq 2$, degree $d \geq 2$ and let $\mathcal{E}$ be a rank $r \geq 2$ Ulrich bundle on X. If $c_1(\mathcal{E})^2 \neq 0$ (hence, in particular, if $\mathcal{E}$ is big), then all Ulrich subvarieties associated to $\mathcal{E}$ are nonempty.

Indeed, note that if $\mathcal{E}$ is big, then $c_1(\mathcal{E})^n > 0$ by [LM, Rmk. 2.2]), hence also $c_1(\mathcal{E})^2 \neq 0$. Now, if there is an empty Ulrich subvariety associated to $\mathcal{E}$, then Theorem 2 implies that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple over a curve, hence $c_1(\mathcal{E})^2 = 0$, a contradiction.

6. Connectedness of degeneracy loci and of Ulrich subvarieties

We study in this section the connectedness of some degeneracy loci associated to a given globally generated bundle. As a particular case, we will get connectedness of Ulrich subvarieties.

Our first observation is that they all have the same number of connected components.

Proposition 6.1. *Let $\mathcal{E}$ be a rank r globally generated bundle on X, let $k \geq 1$ and assume that $c_k(\mathcal{E}) \neq 0$. Consider the set of morphisms $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$ such that the degeneracy locus $D_{r-k}(\varphi)$ is reduced of pure codimension k. Then the above set is open and all such degeneracy loci $D_{r-k}(\varphi)$ have the same number of connected (or irreducible) components. In particular, if $\mathcal{E}$ is Ulrich with $c_2(\mathcal{E}) \neq 0$, then all Ulrich subvarieties associated to $\mathcal{E}$ have the same number of connected components.*

Proof. Let $W = \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X^{\oplus(r+1-k)}, \mathcal{E})$, let $T = \operatorname{Spec}(\mathbb{C}[W^*])$ and consider $X \times T$ with projections $\pi_1 : X \times T \to X, \pi_2 : X \times T \to T$. A (closed) point $t \in T$ corresponds to a morphism $\varphi_t : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$. As is well-known (see for instance the proof of [Ba, Statement (folklore)(i), §4.1]), the degeneracy loci $D_{r-k}(\varphi_t)$, for $\varphi_t \in W$, arise as fibers of a morphism $\pi_{\mathcal{Z}} : \mathcal{Z} \to T$, where $\mathcal{Z} \subset X \times T$ is a certain closed subscheme of codimension k and $\pi_{\mathcal{Z}} = \pi_{2|\mathcal{Z}}$. Since $X \to \operatorname{Spec}(\mathbb{C})$ is proper, then so is π_2 by base extension. In particular $\pi_{\mathcal{Z}}$ is proper as well. We now observe that $\pi_{\mathcal{Z}}$ is surjective. Indeed, we know from Lemmas 4.1 and 3.5, that we can find an open subset $U' \subset T$ such that $Z_t := D_{r-k}(\varphi_t) \subset X$, for $t \in U'$, is reduced of pure codimension k. This means that $U' \subset \pi_{\mathcal{Z}}(\mathcal{Z})$. As T is integral and $\pi_{\mathcal{Z}}$ is closed, we conclude that $\pi_{\mathcal{Z}}(\mathcal{Z}) = T$. Next, let $U \subset T$ be the (integral) open subscheme such that fibers $\pi_{\mathcal{Z}}^{-1}(t) \subset X \times \{t\} \cong X$ have the expected codimension, that is equivalent to say that $t \in U$ if and only if $D_{r-k}(\varphi_t)$ has codimension k. Then the base change $p : \mathcal{Z} \times_T U = \pi_{\mathcal{Z}}^{-1}(U) = \mathcal{Z}_U \to U$ is proper with geometric fibers being the degeneracy loci $Z_t \subset X$ of expected codimension k.

We claim that p is also flat. To see this, let $\mathcal{O}_X(1)$ be a very ample line bundle on X. First observe that, since $L = \pi_1^* \mathcal{O}_X(1)$ is T-ample by [GW1, Prop. 13.64], the restriction $L_{|\mathcal{Z}}$ is still T-ample by [GW1, Rmk. 13.61(2)]. Therefore, again by [GW1, Prop. 13.64], the base change $L_U = (L_{|\mathcal{Z}})_{|\pi_{\mathcal{Z}}^{-1}(U)}$ of $L_{|\mathcal{Z}}$ is U-ample. On the other hand, $(L_U)_{|p^{-1}(t)} = \mathcal{O}_{Z_t}(1)$ for all $t \in U$ and all $p^{-1}(t) = Z_t \subset X \subset \mathbb{P}^N$ have the same Hilbert polynomial with respect to $\mathcal{O}_X(1)$: In fact, their structure sheaf, fits, by the Eagon-Northcott complex, into an exact sequence of the form

$$0 \to \mathcal{O}_X(-D)^{\oplus\binom{r-1}{r-k}} \to \mathcal{E}(-D)^{\oplus\binom{r-2}{r-k}} \to \Lambda^2\mathcal{E}(-D)^{\oplus\binom{r-3}{r-k}} \to \cdots \to \Lambda^{k-1}\mathcal{E}(-D) \to \mathcal{O}_X \to \mathcal{O}_{Z_t} \to 0$$

where $\det \mathcal{E} = \mathcal{O}_X(D)$, hence $\chi(\mathcal{O}_{Z_t}(m))$ is independent of t. Then the claim follows from [GW2, Thm. 23.155]. Since $p : \mathcal{Z}_U \to U$ is proper and flat, the set of points $V \subset U$ such that $p^{-1}(t) = Z_t$ is reduced for $t \in V$ is open (see [G2, Thm. 12.2.1] or [GW1, (E.1)(11)]). This proves the first part of the claim. Taking another base change, we get a proper flat morphism $q : \mathcal{Z}_U \times_U V = p^{-1}(V) \to V$ over an integral scheme such that every geometric fiber $q^{-1}(t) = D_{r-k}(\varphi_t) \subset X$ is reduced of codimension k. Then [ST, Lemma 37.53.8, Tag 0E0N] tells that the number of connected components of fibers is constant. Since the degeneracy loci $D_{r-k}(\varphi)$ under consideration are reduced and have codimension k, they arise as fibers over some $t \in V$ and therefore they have the same number of connected components. In particular this holds for Ulrich subvarieties by their definition and Lemmas 5.1(i) and 4.1. $\square$

We now proceed with the proof of Theorem 1. We first study the case $c_{k+1}(\mathcal{E}) = 0$, in which we can actually say a bit more.

Lemma 6.2. *Let $\mathcal{E}$ be a rank r globally generated bundle on X with $c_k(\mathcal{E}) \neq 0, c_{k+1}(\mathcal{E}) = 0$ for some integer k and let $s = h^0(\mathcal{E}^*)$. Consider morphisms $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$ such that the degeneracy loci $D_{r-k}(\varphi)$ are reduced of pure codimension k. If $k \in \{1, 2\}$, or if $k \geq 3$ and $H^t((\Lambda^{k-1-t}\mathcal{E})(-\det \mathcal{E})) = 0$ for $1 \leq t \leq k-2$, then all degeneracy loci $D_{r-k}(\varphi)$ as above have at least $r+1-k-s$ connected components.*

Proof. Note that $k \leq \min\{r, n\}$ since $c_k(\mathcal{E}) \neq 0$. Let $V_0 \subset H^0(\mathcal{E})$ be a general subspace of dimension $r+1-k$ and let $\varphi_0 : V_0 \otimes \mathcal{O}_X \to \mathcal{E}$. If $r = k$, we set $\mathcal{F} = \mathcal{E}$. If $r \geq k+1$, let $V_1 \subset V_0$ be a general subspace of dimension $r-k$. Consider $\varphi_1 : V_1 \otimes \mathcal{O}_X \to \mathcal{E}$, set $\mathcal{F} = \operatorname{Coker} \varphi_1, \mathcal{H} = \operatorname{Coker} \varphi_0$, so that we have an exact sequence

$$0 \to V_0 \otimes \mathcal{O}_X \to \mathcal{E} \to \mathcal{H} \to 0 \tag{6.1}$$

and a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V_1 \otimes \mathcal{O}_X \arrow[r, "\varphi_1"] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & V_0 \otimes \mathcal{O}_X \arrow[r, "\varphi_0"] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] & 0 \\
 & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Since $c_{k+1}(\mathcal{E}) = 0$, we deduce by Lemma 4.1 when $k \leq n-1$ and by Lemma 3.5 when $k = n$, that $D_{r-k-1}(\varphi_1) = \emptyset$, hence $\mathcal{F}$ is a rank k vector bundle on X and the snake lemma applied to the above diagram gives an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{H} \to 0. \tag{6.2}$$

Note that the above holds also when $r = k$. Let $W = D_{r-k}(\varphi_0)$, so that W is reduced of pure codimension k by Lemmas 4.1 and 3.5. When $r = k$, we have that $V_0 = \langle \sigma \rangle$ and $W = Z(\sigma)$. When $r \geq k+1$, let $\{\sigma_1, \dots, \sigma_{r-k}\}$ be a basis of V_1 and let $\{\sigma, \sigma_1, \dots, \sigma_{r-k}\}$ be a basis of V_0. We claim that

$$W = Z(\alpha(\sigma)) \tag{6.3}$$

where $\alpha = H^0(\psi) : H^0(\mathcal{E}) \to H^0(\mathcal{F})$ is the map induced by the exact sequence

$$0 \to V_1 \otimes \mathcal{O}_X \xrightarrow{\varphi_1} \mathcal{E} \xrightarrow{\psi} \mathcal{F} \to 0. \tag{6.4}$$

In fact, in a given point $x \in X$, we have the commutative diagram of $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$-vector spaces

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & V_1 \otimes \mathcal{O}_X(x) \arrow[r, "\varphi_1(x)"] \arrow[d] & \mathcal{E}(x) \arrow[r, "\psi(x)"] \arrow[d] & \mathcal{F}(x) \arrow[r] & 0 \\
 & V_0 \otimes \mathcal{O}_X(x) \arrow[r, "\varphi_0(x)"] \arrow[d] & \mathcal{E}(x) \arrow[d] & & \\
 & \mathcal{O}_X(x) \arrow[r] \arrow[d] & 0 & & \\
 & 0 & & &
\end{tikzcd}$$

where the first row is exact since $D_{r-k-1}(\varphi_1) = \emptyset$. Note that $\psi(x)(\sigma(x)) = (\alpha(\sigma))(x)$. Now $x \in W$ if and only if $\operatorname{rank} \varphi_0(x) \le r-k$, if and only if $\operatorname{Im}\varphi_0(x) = \operatorname{Im}\varphi_1(x)$, if and only if $\sigma(x) \in \operatorname{Ker}\psi(x)$, if and only if $(\alpha(\sigma))(x) = 0$, if and only if $x \in Z(\alpha(\sigma))$. Thus, (6.3) is proved. Dualizing (6.2), we get the exact sequence

$$0 \to \mathcal{H}^* \to \mathcal{F}^* \to \mathcal{O}_X \to \mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{H}, \mathcal{O}_X) \to 0 \tag{6.5}$$

so that, by (6.3), $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{H}, \mathcal{O}_X) \cong \mathcal{O}_W$. Therefore, dualizing (6.1), we find the exact sequence

$$0 \to \mathcal{H}^* \to \mathcal{E}^* \to V_0^* \otimes \mathcal{O}_X \to \mathcal{O}_W \to 0$$

that splits into the exact sequences

$$0 \to \mathcal{H}^* \to \mathcal{E}^* \to \mathcal{G} \to 0 \tag{6.6}$$

and

$$0 \to \mathcal{G} \to V_0^* \otimes \mathcal{O}_X \to \mathcal{O}_W \to 0. \tag{6.7}$$

Assume, for the time being, that

$$H^1(\mathcal{H}^*) = 0. \tag{6.8}$$

Then we have that $h^0(\mathcal{G}) \le h^0(\mathcal{E}^*) = s$ by (6.6) and (6.8). But then (6.7) gives that

$$r + 1 - k - s \le h^0(V_0^* \otimes \mathcal{O}_X) - h^0(\mathcal{G}) \le h^0(\mathcal{O}_W)$$

and therefore W has at least $r+1-k-s$ connected components. Hence the same holds for all degeneracy loci $D_{r-k}(\varphi)$ that are reduced of pure codimension k by Proposition 6.1.

It remains to show (6.8), for which we distinguish in cases, according to k. Let $\mathcal{O}_X(D) = \det \mathcal{E} = \det \mathcal{F}$. If $k = 1$, we have that $\mathcal{F} = \det \mathcal{E}$ and (6.2) shows that $\mathcal{H} \cong (\det \mathcal{E})_{|W}$. Hence $\mathcal{H}^* = 0$ and (6.8) holds in this case. If $k = 2$, we have that $\dim V_0 = r - 1$, so that $\mathcal{H} \cong \mathcal{J}_{W/X}(D)$ by the Eagon-Northcott resolution. Since $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_{W/X}, \mathcal{O}_X) \cong \mathcal{O}_X$ by [AK, Lemma IV.5.1], we have that $\mathcal{H}^* \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}_{Z/X}(D), \mathcal{O}_X) \cong \mathcal{O}_X(-D)$, and then (6.8) follows from Lemma 4.2(ii). As for the case $k \ge 3$, we will use the hypothesis $H^t((\Lambda^{k-t-1}\mathcal{E})(-D)) = 0$ for $1 \le t \le k-2$ to prove (6.8). To this end, we first prove the ensuing

Claim 6.3. *We have:*

(i) $H^{k-1}(\mathcal{O}_X(-D)) = 0$.
(ii) $H^i((\Lambda^{k-i-1}\mathcal{F})(-D)) = 0$, *for* $1 \le i \le k-2$.

Proof. Since $c_k(\mathcal{E}) \ne 0$, (i) follows from Lemma 4.2(ii). As for (ii), consider, for each $i \in \{1, \dots, k-2\}$, the Eagon-Northcott-type exact sequence associated to (6.4):

$$0 \to F_{k-i-1} \to \cdots \to F_1 \to F_0 \to (\Lambda^{k-i-1}\mathcal{F})(-D) \to 0$$

where $F_j = S^j V_1 \otimes (\Lambda^{k-i-j-1}\mathcal{E})(-D), 0 \le j \le k-i-1$. In order to prove (ii), it will suffice, by [Laz1, Prop. B.1.2(i)]), to show that $H^{i+j}(F_j) = 0$ for all $0 \le j \le k-i-1$. Now, if $0 \le j \le k-i-2$, we have that $H^{i+j}(F_j) = S^jV_1 \otimes H^{i+j}((\Lambda^{k-i-j-1}\mathcal{E})(-D)) = 0$ by assumption. If $j = k-i-1$ we have that $H^{k-1}(F_{k-i-1}) = S^{k-i-1}V_1 \otimes H^{k-1}(\mathcal{O}_X(-D)) = 0$ by (i). This proves Claim 6.3. □

We now continue the proof of the lemma. By (6.3), we have the Koszul resolution

$$0 \to \Lambda^k \mathcal{F}^* \to \Lambda^{k-1}\mathcal{F}^* \to \cdots \to \Lambda^2\mathcal{F}^* \to \mathcal{F}^* \to \mathcal{J}_{W/X} \to 0$$

that, using (6.5) and the fact that $\mathcal{E}xt^1_{\mathcal{O}_X}(\mathcal{H}, \mathcal{O}_X) \cong \mathcal{O}_W$, can be split into

$$0 \to \mathcal{O}_X(-D) \to \Lambda^{k-1}\mathcal{F}^* \to \cdots \to \Lambda^2\mathcal{F}^* \to \mathcal{H}^* \to 0. \tag{6.9}$$

Again by [Laz1, Prop. B.1.2(i)], in order to prove (6.8), we will need to show that $H^i(\Lambda^{i+1}\mathcal{F}^*) = 0$ for $1 \le i \le k-1$. Finally, the latter follows from Claim 6.3 since $H^i(\Lambda^{i+1}\mathcal{F}^*) \cong H^i((\Lambda^{k-i-1}\mathcal{F})(-D))$. This concludes the proof of the lemma. □

Regarding the assumption $H^t((\Lambda^{k-t-1}\mathcal{E})(-\det\mathcal{E})) = 0$ for $1 \le t \le k-2$ in Lemma 6.2, we can find many examples of Ulrich bundles satisfying it.

Lemma 6.4. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension $n \ge 3$ and degree d. Let $\mathcal{E}$ be a rank r Ulrich bundle on X. Let $k \in \mathbb{Z}$ be such that $3 \le k \le r$. Consider the following conditions:*

(i) $\det\mathcal{E} = \mathcal{O}_X(u)$.
(ii) $\det\mathcal{E} = \mathcal{O}_X(u)$ *and $X \subset \mathbb{P}^N$ is subcanonical of degree $d \ge 3$.*
(iii) $\det\mathcal{E} = \mathcal{O}_X(u), n \ge k-1$ *and $X \subset \mathbb{P}^N$ is subcanonical with $K_X = \mathcal{O}_X(-i_X)$ such that $i_X \le n+3-k$ and $\mathcal{O}_X(1)$ is $2n$-Koszul (see [To] for the definition of M-Koszul line bundle).*

Then $H^t((\Lambda^{k-t-1}\mathcal{E})(-\det\mathcal{E})) = 0$ for all $1 \le t \le k-2$ is implied by (i) if $k = 3$, by (ii) if $k = 4$, by (iii) for any $k \ge 5$.

Proof. The condition for $k = 3$ is just $H^1(\mathcal{E}(-u)) = 0$ and it is immediately satisfied because $\mathcal{E}$ is aCM by Lemma 5.1(vi). Now let $k \ge 4$ and suppose that $X \subset \mathbb{P}^N$ is subcanonical with $K_X = \mathcal{O}_X(-i_X)$. Then $H^{k-2}(\mathcal{E}(-u)) = 0$ because $\mathcal{E}$ is aCM by Lemma 5.1(vi) and $n \ge k-1$ in any case. Using Serre duality we get

$$h^t(\Lambda^{k-t-1}\mathcal{E}(-u)) = h^t(\Lambda^{r+t+1-k}\mathcal{E}^*) = h^{n-t}(\Lambda^{r+t+1-k}\mathcal{E}(-i_X)) \tag{6.10}$$

for all $1 \le t \le k-3$. The claim for $k = 4$ follows from [LR3, Lemma 4.2(iii)] since $d \ge 3$, hence $i_X \le n-1$. For $k \ge 5$, consider the assumptions in (iii). Then $\Lambda^q\mathcal{E}$ is still 0-regular for every $q \ge 1$ by [To, Thm. 3.4] as $\mathcal{O}_X(1)$ is $2n$-Koszul. It follows that $H^i((\Lambda^q\mathcal{E})(l)) = 0$ for $i \ge 1, l \ge -i, q \ge 1$. As $-i_X \ge -n-3+k \ge -n+t$ for $1 \le t \le k-3$, we obtain the conclusion by (6.10). □

We point out that, as shown in the proof of Theorem 1 below, to obtain, for any globally generated bundle $\mathcal{E}$, the vanishing $H^1(\mathcal{E}(-\det\mathcal{E})) = 0$, it is enough to suppose that $c_3(\mathcal{E}) \ne 0$ and $H^1(\mathcal{O}_X) = 0$, which is a much weaker assumption than Lemma 6.4(i). Anyway, triples $(X, \mathcal{O}_X(1), \mathcal{E})$ satisfying the conditions in Lemma 6.4 exist. For (i) and (ii), the conditions hold for example if $\operatorname{Pic}(X) \cong \mathbb{Z}\mathcal{O}_X(1)$ and $k = 3$, or $k = 4$ and $d \ge 3$. As for (iii), some examples can be obtained in $(\mathbb{G}(1, r+1), \mathcal{O}_G(1))$ by [CMRPL, Thm. 7.2.5] and [Ra].

When $k = 2$, we can say more.

Lemma 6.5. *In the case $k = 2$, further assume in Lemma 6.2 that $c_1(\mathcal{E})^3 \ne 0$. Let $h = h^1(\mathcal{E}^*)$. Then:*

(a) *If $h = 0$, then the connected components of such $D_{r-2}(\varphi)$'s are exactly $r-s-1$.*
(b) *If $h > 0, s = 0$ and $H^1(\mathcal{O}_X) = 0$, then the connected components of such $D_{r-2}(\varphi)$'s are exactly $r+h-1$. Moreover there exists a globally generated vector bundle $\tilde{\mathcal{E}}$ of rank $r+h$ on X fitting into a non-split exact sequence*

$$0 \to \mathcal{O}_X^{\oplus h} \to \tilde{\mathcal{E}} \to \mathcal{E} \to 0$$

with $H^0(\tilde{\mathcal{E}}^*) = H^1(\tilde{\mathcal{E}}^*) = 0$ and such that all reduced degeneracy loci $D_{r+h-2}(\tilde{\varphi})$ of pure codimension 2 for injective morphisms $\tilde{\varphi} : \mathcal{O}_X^{\oplus(r+h-1)} \to \tilde{\mathcal{E}}$ have exactly $r+h-1$ connected components. In addition to this, for any fixed such $W' = D_{r-2}(\varphi')$, there exists an injective morphism $\tilde{\varphi}' : \mathcal{O}_X^{\oplus(r+h-1)} \to \tilde{\mathcal{E}}$ such that $D_{r+h-2}(\tilde{\varphi}') = W'$.
(c) *If $r = 2$ and $s = 0$, then $H^1(\mathcal{E}(-D)) \cong H^1(\mathcal{E}^*) \cong H^1(\mathcal{J}_{D_{r-2}(\varphi)/X})$ for any such degeneracy loci $D_{r-2}(\varphi)$.*

Proof. The assumption $c_1(\mathcal{E})^3 \neq 0$ gives $H^j(\mathcal{O}_X(-D)) = 0$ for $0 \le j \le 2$ by Lemma 4.2(ii). Also we have that $\mathcal{H}^* \cong \mathcal{O}_X(-D)$. To see (a), since $H^1(\mathcal{E}^*) = 0$, we get from (6.6) that $h^0(\mathcal{G}) = s$ and $H^1(\mathcal{G}) = 0$. Therefore, (6.7) implies that $h^0(\mathcal{O}_W) = h^0(V_0^* \otimes \mathcal{O}_X) = r - s - 1$, as claimed.

As for (b), (6.6) yields $h^0(\mathcal{G}) = 0$ and $H^1(\mathcal{G}) \cong H^1(\mathcal{E}^*)$. Therefore, by (6.7), we obtain $h^0(\mathcal{O}_W) = r + h - 1$ as desired. Now, take $\Xi = H^1(\mathcal{E}^*)$ and suppose that $h = dim(\Xi) > 0$. Since

$$\mathrm{Id}_\Xi \in Hom(\Xi, \Xi) \cong \Xi^* \otimes \Xi \cong \mathrm{Ext}^1(\mathcal{E}, \Xi^* \otimes \mathcal{O}_X)$$

this gives a non-split exact sequence of vector bundles

$$0 \to \Xi^* \otimes \mathcal{O}_X \xrightarrow{\xi} \tilde{\mathcal{E}} \to \mathcal{E} \to 0 \tag{6.11}$$

with $\tilde{\mathcal{E}}$ of rank $r + h$. It follows that $c_i(\tilde{\mathcal{E}}) = c_i(\mathcal{E})$ for all $i \ge 1$ and that $\tilde{\mathcal{E}}$ is globally generated since $\mathcal{E}$ is and $H^1(\mathcal{O}_X) = 0$. Dualizing (6.11) and taking cohomology we get, using $H^0(\mathcal{E}^*) = 0$, an exact sequence

$$0 \to H^0(\tilde{\mathcal{E}}^*) \to \Xi \xrightarrow{\delta} \Xi \to H^1(\tilde{\mathcal{E}}^*) \to 0$$

where, by construction, $\delta = \mathrm{Id}_\Xi$. It follows that $h^0(\tilde{\mathcal{E}}^*) = h^1(\tilde{\mathcal{E}}^*) = 0$. Now, the map on global sections $H^0(\tilde{\mathcal{E}}) \xrightarrow{\alpha} H^0(\mathcal{E})$ is surjective as $H^1(\mathcal{O}_X) = 0$. Let $V' \subset H^0(\mathcal{E})$ with $\dim V' = r - 1$ and such that $W' := D_{r-2}(\varphi')$ is reduced of pure codimension 2, where $\varphi' : V' \otimes \mathcal{O}_X \to \mathcal{E}$. Set $\tilde{V}' = \alpha^{-1}(V') \subset H^0(\tilde{\mathcal{E}})$. Then $\tilde{V}' \cong \Xi^* \oplus V'$ and there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Xi^* \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\mathrm{Id}_{\Xi^*}"] & \tilde{V}' \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\tilde{\varphi}'"] & V' \otimes \mathcal{O}_X \arrow[r] \arrow[d, "\varphi'"] & 0 \\
0 \arrow[r] & \Xi^* \otimes \mathcal{O}_X \arrow[r, "\xi"] & \tilde{\mathcal{E}} \arrow[r] & \mathcal{E} \arrow[r] & 0.
\end{tikzcd}$$

To show that $D_{r+h-2}(\tilde{\varphi}') = W'$, observe that, since ξ never drops rank, the second row remains exact after tensoring with $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ for any point $x \in X$. Hence we have $\tilde{\mathcal{E}}(x) \cong \Xi^*(x) \oplus \mathcal{E}(x)$, where $\Xi^*(x) \cong (\Xi^* \otimes \mathcal{O}_X)(x)$ is the isomorphic image of $\xi(x)$. Therefore $\tilde{\varphi}'(x) = \xi(x) \oplus \varphi'(x)$. Since $\xi(x)$ has maximal rank, this says that $\tilde{\varphi}'(x)$ drops rank if and only if $\varphi'(x)$ does, which means that $D_{r+h-2}(\tilde{\varphi}')$ and $D_{r-2}(\varphi')$ coincide as subschemes of X, because they are locally defined by the vanishing of the same minors (see Remark 3.4). By part (a) we get (b). Finally, to prove (c), observe that (6.1) twisted by $\mathcal{O}_X(-D)$ reads

$$0 \to \mathcal{O}_X(-D) \to \mathcal{E}^* \to \mathcal{J}_{W/X} \to 0$$

where we used $\mathcal{E}^* \cong \mathcal{E}(-D)$. Taking cohomology and recalling that $H^j(\mathcal{O}_X(-D)) = 0$ for $1 \le j \le 2$, we get the claim. $\square$

For $k = 3$ we have.

Lemma 6.6. *In the case $k = 3$, further assume in Lemma 6.2 that $X \subset \mathbb{P}^N$ is subcanonical with $n \ge 4$ and that $\mathcal{E}$ is aCM with $\det \mathcal{E} = \mathcal{O}_X(u), u > 0$. Then the connected components of such $D_{r-3}(\varphi)$'s are exactly $r - s - 2$.*

Proof. Since $\mathcal{E}$ is aCM and X is subcanonical, we have that $H^i(\mathcal{E}(-u)) = 0$ for $1 \le i \le 2$ and $H^1(\mathcal{E}^*) \cong H^{n-1}(\mathcal{E}(-i_X)) = 0$. Now, $c_3(\mathcal{E}) \neq 0$, hence also $c_2(\mathcal{E}) \neq 0$ by Lemma 4.1. Choosing a general subspace $V \subset H^0(\mathcal{E})$ of dimension $r - 1$, we get that (4.4) holds with $Z \neq \emptyset$ by Lemma 4.1. Hence (4.4) implies that $H^0(\mathcal{E}(-u)) = 0$. Also, $H^i(\mathcal{O}_X(-u)) = 0$ for $1 \le i \le 3$ by Kodaira vanishing. Using the exact sequence (6.4) we deduce that $H^i(\mathcal{F}(-u)) = 0$ for $0 \le i \le 2$. Since $\mathcal{F}$ has rank 3, (6.9) becomes

$$0 \to \mathcal{O}_X(-u) \to \mathcal{F}(-u) \to \mathcal{H}^* \to 0$$

and we find that $H^i(\mathcal{H}^*) = 0$ for $0 \le i \le 2$. Now, (6.6) gives that $H^1(\mathcal{G}) = 0$ and $h^0(\mathcal{G}) = s$ and then (6.7) implies that $h^0(\mathcal{O}_W) = r - s - 2$ as required. $\square$

We now prove Theorem 1.

Proof of Theorem 1. Let H be a very ample divisor on X. We have that $r \ge k + s$ by Lemma 4.2(iii) and $k \le n$ since $c_k(\mathcal{E}) \neq 0$.

First, assume that $k \in \{1, 2\}$. The fact that (i) implies (ii) is the content of Lemma 6.2. Now assume (iii). If $k = n$ we have obviously $c_{k+1}(\mathcal{E}) = 0$, hence we can assume that $k \le n - 1$. Let $V_0 \subset H^0(\mathcal{E})$ be a general subspace of dimension $r+1-k$ and let $\varphi_0 : V_0 \otimes \mathcal{O}_X \to \mathcal{E}$. It follows from Lemmas 4.1 and 3.5 that $Z := D_{r-k}(\varphi_0)$ is reduced of pure codimension k and Z is not connected by hypothesis.

If $n = k+1$, we set $X' = X, \mathcal{E}' = \mathcal{E}$ and $Z' = Z$. If $n \geq k+2$, cutting down with $n-k-1$ general $H_1, \dots, H_{n-k-1} \in |H|$, we get a smooth $(k+1)$-fold X' and a globally generated bundle $\mathcal{E}' = \mathcal{E}_{|X'}$ with $c_k(\mathcal{E}') \neq 0$ by Lemma 4.1(ii). Moreover, observe that

$$Z \cap H_1 \cap \ldots \cap H_{n-k-1} = Z \cap X' = D_{r-k}(\varphi_0) \cap X' = D_{r-k}(\varphi_{0|X'})$$

is reduced of pure codimension k and disconnected. Let $V' \subset H^0(\mathcal{E}')$ be a general subspace of dimension $r+1-k$, let $\varphi' : V' \otimes \mathcal{O}_{X'} \to \mathcal{E}'$, so that $Z' := D_{r-k}(\varphi')$ is reduced of pure codimension k by Lemmas 4.1 and 3.5. Also, Z' is disconnected by Proposition 6.1.

If $k=1$, we have that X' is a smooth surface and $Z' \in |\det \mathcal{E}'|$ is disconnected, hence, since $\det \mathcal{E}'$ is globally generated, it follows from Bertini's theorem that $c_1(\mathcal{E}')^2 = 0$, and therefore $c_2(\mathcal{E}) = 0$ by Lemmas 4.2(i) and 4.1(ii).

If $k=2$, Lemma 4.5 gives a smooth irreducible surface $Y' \in |\det \mathcal{E}'|$ containing Z', with $\mathcal{O}_{Y'}(Z')$ globally generated. Since $Z' \in |\mathcal{O}_{Y'}(Z')|$ is disconnected, Bertini's theorem implies that $|\mathcal{O}_{Y'}(Z')|$ is composite with a pencil, and therefore $(Z')^2 = 0$. Then, (4.3) gives that $c_3(\mathcal{E})H^{n-3} = c_3(\mathcal{E}') = (Z')^2 = 0$, hence $c_3(\mathcal{E}) = 0$ by Lemma 4.1(ii).

Thus, (i) is proved in both cases $k \in \{1,2\}$. Now, if $r \geq k+s+1$, then clearly (ii) implies (iii) and therefore, if $r \geq k+s+1$, we have that (i), (ii) and (iii) are equivalent.

Next, we show that (i) implies (ii) when $k=3$ and $H^1(\mathcal{O}_X) = 0$. To this end, setting $\mathcal{O}_X(D) = \det \mathcal{E}$, it is enough to prove by Lemma 6.2 that

$$H^1(\mathcal{E}(-D)) = 0. \tag{6.12}$$

Let $V \subset H^0(\mathcal{E})$ be a general subspace of dimension $r-1$ and let $Z = D_{r-2}(\varphi)$, where $\varphi : V \otimes \mathcal{O}_X \to \mathcal{E}$. Since $c_3(\mathcal{E}) \neq 0$ by hypothesis, it follows from Lemma 4.1 that $c_2(\mathcal{E}) \neq 0$. Hence Z is connected, since for $k=2$ we have already proved that (iii) implies (i). Also, Lemma 4.2(ii) gives that $H^1(\mathcal{O}_X(-D)) = 0$. Since $H^1(\mathcal{O}_X) = 0$, the exact sequence

$$0 \to \mathcal{J}_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

shows that $H^1(\mathcal{J}_{Z/X}) = 0$. Then (6.12) follows from the exact sequence

$$0 \to V \otimes \mathcal{O}_X(-D) \to \mathcal{E}(-D) \to \mathcal{J}_{Z/X} \to 0.$$

Now, assume again that $k \in \{1,2\}$. To see (iv), observe that if $D_{r-k}(\varphi)$ is disconnected, then (iii) holds and therefore so does (i), contradicting Proposition 4.3(i). As for (v), assume that $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$ is a general morphism and that $D_{r-k}(\varphi)$ is singular. It follows from Lemma 3.5, that $D_{r-k-1}(\varphi) = \mathrm{Sing}(D_{r-k}(\varphi)) \neq \emptyset$ of the expected codimension $2k+2$. Then $[D_{r-k-1}(\varphi)] = c_{k+1}(\mathcal{E})^2 - c_k(\mathcal{E})c_{k+2}(\mathcal{E})$ by Porteous' formula (see for example [EH, Thm. 12.4]). Hence, if $D_{r-k}(\varphi)$ were disconnected, we would have, by Theorem 1, that $c_{k+1}(\mathcal{E}) = 0$ and also $c_{k+2}(\mathcal{E}) = 0$ by Lemma 4.1, hence the contradiction $[D_{r-k-1}(\varphi)] = 0$. □

The following simple example shows that Theorem 1 (or any possible generalization to $k \geq 3$) is sharp regarding the inequality $r \geq s+k+1$.

Remark 6.7. Let $1 \leq k \leq \min\{r,n\}$, let $\mathcal{G} = \mathcal{O}_{\mathbb{P}^k}(1)^{\oplus k} \oplus \mathcal{O}_{\mathbb{P}^k}^{\oplus(r-k)}$, let $X = \mathbb{P}^{n-k} \times \mathbb{P}^k$ and let $\mathcal{E} = \pi^*\mathcal{G}$, where $\pi : \mathbb{P}^{n-k} \times \mathbb{P}^k \to \mathbb{P}^k$ is the second projection. Then we have that $\mathcal{E}$ is globally generated, $s = h^0(\mathcal{E}^*) = r-k$ and, for any $t \in \mathbb{P}^k$, $c_k(\mathcal{E}) = [\mathbb{P}^{n-k} \times \{t\}] \neq 0, c_{k+1}(\mathcal{E}) = 0$. Moreover, if $\varphi : \mathcal{O}_X^{\oplus(r+1-k)} \to \mathcal{E}$ is general, then $D_{r-k}(\mathcal{E}) = \mathbb{P}^{n-k} \times \{t\}$ is connected.

Next, we prove Corollary 1.

Proof of Corollary 1. We have that $\mathcal{E}$ is globally generated by Lemma 5.1(i) and $h^0(\mathcal{E}^*) = 0$, for otherwise Lemma 5.1(iv) gives the contradiction $c_2(\mathcal{E}) = 0$. Hence Theorem 1 with $k=2$ applies to Ulrich subvarieties by their same definition and Lemma 4.1. Thus, we get (v) and the equivalence of (i), (ii) and (iii). As for (iv), by [Bu, Thm. 2] we know that

$$\mathbf{B}_+(\mathcal{E}) = \bigcup_L L \tag{6.13}$$

where L ranges over all lines $L \subset X$ such that $\mathcal{E}_{|L}$ is not ample. The assumption in (iv) and (6.13) say exactly that $\mathbf{B}_+(\mathcal{E}) \neq X$, therefore (iv) follows from Theorem 1(iv). $\square$

Remark 6.8. Let $n \geq 3, r \geq 3$, and let $\mathcal{E}$ be a rank r Ulrich bundle on $X \subset \mathbb{P}^N$. If X is not covered by lines, then all Ulrich subvarieties associated to $\mathcal{E}$ are connected by Corollary 1(iv). However, Ulrich subvarieties can be connected even if X is covered by lines. For instance, there are varieties $X \subset \mathbb{P}^N$ covered by lines supporting very ample Ulrich bundles $\mathcal{E}$ (see [LS, Rmk. 4.3(i)]). Then $\mathcal{E}_{|L}$ is very ample on all lines $L \subset X$, whence Corollary 1(iv) applies.

Remark 6.9. The converse of Theorem 1(iv) does not hold. For instance, as we will see in Proposition 9.1(i), all non-big Ulrich bundles $\mathcal{E}$ on threefolds $X \subset \mathbb{P}^N$, thus having $\mathbf{B}_+(\mathcal{E}) = X$, which satisfy $c_1(\mathcal{E})^3 > 0$, have connected associated Ulrich subvarieties. Other examples are quadrics $Q_n \subset \mathbb{P}^{n+1}$ for $n \geq 3$ with $\mathcal{E}$ being any Ulrich bundle: Indeed all Ulrich subvarieties associated to $\mathcal{E}$ are connected by Lemma 7.6(ii), but $\mathbf{B}_+(\mathcal{E}) = Q_n$ by [O1, Cor. 1.6] and [Bu, Thm. 2], because all Ulrich bundles are direct sum of spinor bundles.

Remark 6.10. If $\mathcal{E}$ is Ulrich with $c_2(\mathcal{E}) \neq 0, c_3(\mathcal{E}) = 0$ and $X \subset \mathbb{P}^N$ is subcanonical of dimension $n \geq 4$, then all Ulrich subvarieties associated to $\mathcal{E}$ have exactly $r-1$ connected components, unless $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), \pi^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r})$, where $\pi : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ is a projection, and, in the latter case, all Ulrich subvarieties have exactly $2r(r-1)$ connected components. Indeed, observe that $s = h^0(\mathcal{E}^*) = 0$ by Lemma 5.1(iv) and $h^1(\mathcal{E}^*) = h^{n-1}(\mathcal{E}(K_X)) = 0$ by Lemma 5.1(vi). Hence, if $c_1(\mathcal{E})^3 \neq 0$, Lemma 6.5 applies. If $c_1(\mathcal{E})^3 = 0$, we can repeat the proof of [LS, Cor. 4] using now the fact that $c_1(\mathcal{E})^3 = 0$ and that $3 \leq \lfloor \frac{n}{2} + 1 \rfloor$. It follows from that proof that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple over a surface, because $c_2(\mathcal{E}) \neq 0$. On the other hand, X is subcanonical, hence the only possibility is that $K_X = -(n-1)H$, so that X is a del Pezzo manifold. Since $\rho(X) \geq 2$, by the classification of del Pezzo manifolds (see for example [LP, pages 860-861], [F, Table, page 710]), we deduce the only possible case $(X, \mathcal{O}_X(1)) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$. Then, it follows from [LS, Cor. 4.9] that $\mathcal{E} = \pi^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus r})$. Therefore Lemma 4.6 implies that general Ulrich subvarieties associated to $\mathcal{E}$ are a disjoint union of $2r(r-1) = c_2(\mathcal{O}_{\mathbb{P}^2}(2)^{\oplus r})$ planes. Hence all Ulrich subvarieties have exactly $2r(r-1)$ connected components by Proposition 6.1.

7. Some connectedness statements and examples

A special case in which degeneracy loci $D_{r-n}(\varphi)$ associated to $\mathcal{E}$ are connected is when $c_n(\mathcal{E}) = 1$. We observe a few things about this.

Remark 7.1. (We thank F. Moretti for observing that we could use [M, Prop. 1.5].) Let $\mathcal{E}$ be a rank r globally generated bundle on X with $c_n(\mathcal{E}) = 1$. Then X is rational. Indeed, since $c_n(\mathcal{E}) = 1$, we have that $r \geq n$. Now, if $r = n$, set $\mathcal{F} = \mathcal{E}$. If $r > n$, consider a general morphism $\varphi : \mathcal{O}_X^{\oplus(r-n)} \to \mathcal{E}$. It follows from Lemma 3.5 that $D_{r-n-1}(\varphi) = \emptyset$, hence we have an exact sequence

$$0 \to \mathcal{O}_X^{\oplus(r-n)} \to \mathcal{E} \to \mathcal{F} \to 0$$

where $\mathcal{F}$ is a rank n globally generated bundle with $c_n(\mathcal{F}) = c_n(\mathcal{E}) = 1$. Now, in any case, $H^0(\mathcal{F}^*) = 0$ by Lemma 4.2(ii). Note that $h^0(\mathcal{F}) \geq n+1$: if $h^0(\mathcal{F}) = n$, then $\mathcal{F} \cong \mathcal{O}_X^{\oplus n}$ and therefore $c_n(\mathcal{F}) = 0$, a contradiction. Let $W \subseteq H^0(\mathcal{F})$ be a general subspace of dimension $n+1$, so that W generates $\mathcal{F}$ in codimension 2, that is away from $D_{n-1}(\varphi_W)$, where $\varphi_W : W \otimes \mathcal{O}_X \to \mathcal{F}$, by Lemma 3.5. Moreover, if $\sigma \in W$ is a general section, that is a general section in $H^0(\mathcal{F})$, we have that $Z(\sigma)$ is 0-dimensional of degree $c_n(\mathcal{F}) = 1$. Therefore [M, Prop. 1.5] gives that X is rational.

In the case of Ulrich bundles, while the case $n = 2$ is treated in Theorem 3, here we give a few examples in rank n and we prove that, in many cases, $c_n(\mathcal{E}) = 1$ cannot happen in rank $r > n$.

Remark 7.2. Let $X \subset \mathbb{P}^{2n}$ be the rational normal scroll $\varphi_\xi(\mathbb{P}(\mathcal{F}))$, where $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)} \oplus \mathcal{O}_{\mathbb{P}^1}(2)$ and ξ is the tautological line bundle. We have that $\mathcal{L} = \xi - \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$ is Ulrich on X (see Example 10.1) and therefore so is $\mathcal{E} = \mathcal{L}^{\oplus n}$ and $c_n(\mathcal{E}) = \mathcal{L}^n = 1$. Case (iii) of Theorem 3 corresponds to $n = 2$.

On quadrics we have

Lemma 7.3. *Let $n \geq 2$ and let $Q_n \subset \mathbb{P}^{n+1}$ be a smooth quadric. The only rank n Ulrich bundles $\mathcal{E}$ on Q_n with $c_n(\mathcal{E}) = 1$ are the following:*

(i) $n = 2, \mathcal{E} = \mathcal{S}' \oplus \mathcal{S}''$.
(ii) $n = 4, \mathcal{E} = (\mathcal{S}')^{\oplus 2}$ *or* $(\mathcal{S}'')^{\oplus 2}$.
(iii) $n = 8, \mathcal{E} = \mathcal{S}'$ *or* $\mathcal{S}''$.

Proof. Since every Ulrich bundle on Q_n is direct sum of spinor bundles (see for example [BGS, Rmk. 2.5(4)]), that have rank $2^{\lfloor \frac{n-1}{2} \rfloor}$, we have that n is a multiple of $2^{\lfloor \frac{n-1}{2} \rfloor}$, and it follows easily that $n \in \{2, 4, 8\}$. Using [O1, Rmk. 2.9], we obtain that the only cases are the ones listed. Case (i) of Theorem 3 corresponds to $n = 2$. □

Remark 7.4. Assume that $X \subset \mathbb{P}^N$ is not covered by lines. Let $\mathcal{E}$ be a 0-regular (in particular Ulrich) rank r vector bundle with $c_n(\mathcal{E}) = 1$. Then $r = n$. Indeed, $n \leq r$ since $c_n(\mathcal{E}) \neq 0$ and it follows from [Lo, Rmk. 7.3], that $c_n(\mathcal{E}) \geq \binom{r}{n} > 1$ if $r > n$.

In the next two lemmas we prove some connectedness statements. We set $X \subset \mathbb{P}^N$ to be a smooth irreducible variety of dimension n, $\mathcal{E}$ a rank r globally generated bundle on X with $c_2(\mathcal{E}) \neq 0$, $\det \mathcal{E} = \mathcal{O}_X(D)$ and Z a normal pure codimension 2 degeneracy locus $D_{r-2}(\varphi)$, where $\varphi : \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E}$ is an injective morphism.

Lemma 7.5. *If $\mathcal{E}$ is $(n-3)$-ample (equivalently, when $\mathcal{E}$ is Ulrich, if either $X \subset \mathbb{P}^N$ does not contain a linear space of dimension $n-2$, or $\mathcal{E}_{|M}$ does not have a trivial direct summand for every linear space $M \subset X$ of dimension $n-2$), then Z is connected.*

Proof. The equivalence is the content of [LR2, Thm. 1]. Connectedness follows from [Tu, Thm. 6.4(a)]. □

Lemma 7.6. *Assume that $n \geq 3$. Then Z is connected if one of the following holds:*

(i) $c_1(\mathcal{E})^3 \neq 0$ *and* $H^1(\mathcal{E}(-D)) = 0$.
(ii) $\mathcal{E}$ *is Ulrich and* $\operatorname{Pic}(X) \cong \mathbb{Z}A$, *with A ample such that* $h^0(A) \geq 2$.
(iii) $\mathcal{E}$ *is Ulrich, $D - K_X - (n+1)H$ is ample and $r \leq n-1$, where H is very ample (for example when $\mathcal{E}$ is special and $4 \leq r \leq n-1, r$ even).*

Moreover, if $H^1(\mathcal{O}_X) = 0$ and Z is connected, then $H^1(\mathcal{E}(-D)) = 0$.

Proof. If (i) holds, we have that $H^2(\mathcal{O}_X(-D)) = 0$ by Lemma 4.2(ii), hence $H^1(\mathcal{J}_{Z/X}) = 0$ by the Eagon-Northcott resolution

$$0 \to \mathcal{O}_X(-D)^{\oplus(r-1)} \to \mathcal{E}(-D) \to \mathcal{J}_{Z/X} \to 0. \tag{7.1}$$

Therefore, the exact sequence

$$0 \to \mathcal{J}_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0 \tag{7.2}$$

shows that $h^0(\mathcal{O}_Z) = h^0(\mathcal{O}_X) = 1$ and Z is connected. To see (ii), note that $(X, \mathcal{O}_X(1)) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$, for otherwise $\mathcal{E} = \mathcal{O}_{\mathbb{P}^n}^{\oplus r}$ by Lemma 5.1(viii) and then $c_2(\mathcal{E}) = 0$, a contradiction. Setting $K_X = -i_X A$, we deduce, as is well-known, that $i_X \leq n$. Next, we can write $D = aA$ with $a > 0$ by Lemma 5.1(iii). Hence $c_1(\mathcal{E})^n > 0$. Let $H = hA$, so that, if $h = 1$, we have that $H^1(\mathcal{E}(-D)) = H^1(\mathcal{E}(-a)) = 0$ by Lemma 5.1(vi). Now assume that $h \geq 2$, so that

$$\left(n + 1 - \frac{4}{r}\right) h \geq i_X \tag{7.3}$$

holds, since $i_X \leq n$ and $r \geq 2$, because $c_2(\mathcal{E}) \neq 0$. It follows from Lemma 5.1(x) that $a = \frac{r}{2}((n+1)h - i_X)$ and (7.3) implies that $a = \frac{r}{2}((n+1)h - i_X) \geq 2h$. But then $H^1(\mathcal{E}(-D)) = 0$ by Lemma 3.1(ii). Therefore (i) holds and Z is connected by (i). If (iii) holds, we have that $D = K_X + (n+1)H + A$ for an ample A, hence, as $K_X + (n+1)H$ is nef, $D^n > 0$. Let $\mathcal{F} = \mathcal{E}^*(K_X + (n+1)H)$ be the dual Ulrich bundle, as in Lemma 5.1(ix). We have, by Serre's duality

$$h^1(\mathcal{E}(-D)) = h^{n-1}(\omega_X \otimes \mathcal{E}^*(D)) = h^{n-1}(\omega_X \otimes \mathcal{F}(D - K_X - (n+1)H)) = 0$$

by [Laz2, Ex. 7.3.17]. Thus, (iii) follows from (i). Finally, if $H^1(\mathcal{O}_X) = 0$ and Z is connected, (7.2) shows that $H^1(\mathcal{J}_{Z/X}) = 0$. Since $c_2(\mathcal{E}) \neq 0$, we have that $H^1(\mathcal{O}_X(-D)) = 0$ by Lemma 4.2(ii) and (7.1) gives $H^1(\mathcal{E}(-D)) = 0$. □

In the following standard examples we also have connectedness.

Lemma 7.7. *Let B be a smooth irreducible curve and let $\mathcal{F}$ be a very ample rank $n \ge 3$ vector bundle on B. Let $\pi : X = \mathbb{P}(\mathcal{F}) \to B$ and $H \in |\mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)|$. Let M be a line bundle on B and let $\mathcal{G}$ be a vector bundle on B such that $H^i(M) = H^i(\mathcal{G}) = 0$ for every $i \ge 0$. Let $\mathcal{E}$ be a rank r vector bundle that is an extension of type*

$$0 \to \Omega_{X/B}(2H + \pi^* M) \to \mathcal{E} \to \pi^*(\mathcal{G}(\det \mathcal{F})) \to 0.$$

Then $\mathcal{E}$ is an Ulrich bundle and $H^1(\mathcal{E}(-D)) = 0$, where $\det \mathcal{E} = \mathcal{O}_X(D)$. In particular any Ulrich subvariety associated to $\mathcal{E}$ is connected.

Proof. Note that $\Omega_{X/B}(2H + \pi^* M)$ is Ulrich by [LM, Lemma 4.1] and $\pi^*(\mathcal{G}(\det \mathcal{F}))$ is Ulrich by [Lo, Lemma 4.1]. Therefore also $\mathcal{E}$ is Ulrich. Also, $c_1(\mathcal{E})^n > 0$ by [LM, Lemma 4.1] and Lemma 5.1(iii). Now, the last assertion follows from the first and Lemma 7.6(i). Next, we prove that $H^1(\mathcal{E}(-D)) = 0$. Note that $r \ge n-1$. We have that $D = (n-2)H + \pi^* N$, where $N = (n-1)M + \det \mathcal{G} + (r-n+2) \det \mathcal{F}$. Now

$$H^1((\pi^*(\mathcal{G}(\det \mathcal{F})))(-D)) = H^1((\pi^*(\mathcal{G}(\det \mathcal{F} - N)))(2-n)H)) = 0$$

since $R^j \pi_*(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(2-n)) = 0$ for every $j \ge 0$. Also, we have

$$\Omega_{X/B}(2H + \pi^* M)(-D)) = \Omega_{X/B}((4-n)H + \pi^*(M - N))$$

and it remains to show that

$$H^1(\Omega_{X/B}((4-n)H + \pi^*(M-N))) = 0. \tag{7.4}$$

If $n = 3$ we have $R^j \pi_*(\Omega_{X/B}(H + \pi^*(M-N))) = 0$ for every $j \ge 0$ by [Laz2, Lemma 7.3.11(i), (iii)], hence (7.4) follows. If $n \ge 4$ we use the exact sequence

$$0 \to \Omega_{X/B}((4-n)H + \pi^*(M-N)) \to (\pi^* \mathcal{F})((3-n)H + \pi^*(M-N)) \to (4-n)H + \pi^*(M-N) \to 0. \tag{7.5}$$

Since $R^j \pi_*((\pi^* \mathcal{F})((3-n)H + \pi^*(M-N))) = 0$ for every $j \ge 0$, we get that

$$H^1((\pi^* \mathcal{F})((3-n)H + \pi^*(M-N))) = 0.$$

Hence, using (7.5), to see (7.4) and conclude the proof, it remains to show that

$$H^0((4-n)H + \pi^*(M-N)) = 0. \tag{7.6}$$

If $n \ge 5$ we have that $H^0(-H + \pi^*(M-N)) = H^0((\pi_*(-H))(M-N)) = 0$, that is (7.6) holds. If $n = 4$, to see that (7.6), we will show that $\deg(M-N) < 0$. Now

$$M - N = -2M - \det(\mathcal{G}) - (r-2) \det \mathcal{F}.$$

If g is the genus of B, we have from $H^i(\mathcal{G}) = 0$ for every $i \ge 0$ that $\deg \mathcal{G} = (r-3)(g-1)$, hence

$$\deg(M-N) = -(r-1)(g-1) - (r-2) \deg \mathcal{F}.$$

Since $r \ge 3$ and $\deg \mathcal{F} > 0$, we see that $\deg(M-N) < 0$ if $g \ge 1$. On the other hand, if $g = 0$ we have, as $\mathcal{F}$ is very ample rank 4 on $\mathbb{P}^1$, that $\deg \mathcal{F} \ge 4$, hence $\deg(M-N) = r - 1 - (r-2) \deg \mathcal{F} \le -3r + 7 < 0$. This proves (7.6) and the lemma. ☐

On the other hand, many times, Ulrich subvarieties will be disconnected.

Remark 7.8. Note that, when $c_2(\mathcal{E}) \ne 0, c_1(\mathcal{E})^3 = 0$ and $n \ge 4$, then all Ulrich subvarieties are disconnected. In fact, we can repeat the proof of [LS, Cor. 4] using now the fact that $c_1(\mathcal{E})^3 = 0$ and that $3 \le \lfloor \frac{n}{2} + 1 \rfloor$. It follows from that proof that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple over a surface, because $c_2(\mathcal{E}) \ne 0$. Then Lemma 7.9 below shows that all Ulrich subvarieties are disconnected.

Lemma 7.9. *Let $(X, \mathcal{O}_X(1), \mathcal{E})$ be a linear Ulrich triple of dimension $n \ge 3$ over a smooth surface B. Then the Ulrich subvarieties associated to $\mathcal{E}$ are connected if and only if $(X, \mathcal{O}_X(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), q^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}))$, where $q : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the second projection.*

Proof. Consider the case $(X, \mathcal{O}_X(1), \mathcal{E}) \cong (\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), q^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}))$, that is a linear Ulrich triple over $\mathbb{P}^2$ by [Lo, Lemma 4.1] and $c_2(q^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})) = q^* H^2_{\mathbb{P}^2} \neq 0$. Hence, for a general Ulrich subvariety Z associated to $\mathcal{E} = q^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$, we have that $Z \neq \emptyset$ by Proposition 2. Therefore, Lemma 4.6 gives that $\mathcal{O}_Z \cong \pi^* \mathcal{O}_{Z_{\mathbb{P}^2}}$, where $Z_{\mathbb{P}^2} = D_{r-2}(\varphi_{\mathbb{P}^2})$ is 0-dimensional. But then $[Z_{\mathbb{P}^2}] = c_2(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}) = H^2_{\mathbb{P}^2}$ and we get that $Z_{\mathbb{P}^2} = \{P\}$ for some point $P \in \mathbb{P}^2$, hence Z is connected and so are all Ulrich subvarieties by Proposition 6.1.

Vice versa, suppose that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple of dimension $n \geq 3$ over a surface B such that all Ulrich subvarieties associated to $\mathcal{E} = \pi^*(\mathcal{G}(\det \mathcal{F}))$ are connected, where $\pi : X \cong \mathbb{P}(\mathcal{F}) \to B$. We first prove that $(X, \mathcal{O}_X(1), \mathcal{E})$ is not a linear Ulrich triple over a smooth curve B_1. In fact, assume that $(X, \mathcal{O}_X(1), \mathcal{E}) \cong (\mathbb{P}(\mathcal{F}_1), \mathcal{O}_{\mathbb{P}(\mathcal{F}_1)}(1), \pi_1^*(\mathcal{G}_1(\det \mathcal{F}_1))$, where $\pi_1 : X \cong \mathbb{P}(\mathcal{F}_1) \to B_1$. Now, for any fiber F_1 of π_1, we have a morphism $\pi_{|F_1} : \mathbb{P}^{n-1} \cong F_1 \to B$ that is not constant, since the fibers of π are $(n-2)$-dimensional. Hence $\pi_{|F_1}$ is finite-to-one onto its image and we get that $n-1 \leq \dim B = 2$, so that $n = 3$. Hence the fibers F of π are lines and the fibers F_1 of π_1 are planes in $\mathbb{P}^N = \mathbb{P}H^0(\mathcal{O}_X(1))$. Moreover it cannot be that $F \subset F_1$, for otherwise $\pi_{|F_1}$ would contract F. Therefore $F \cap F_1$ is a point and $\pi_{|F_1}$ is an isomorphism. Hence $B \cong \mathbb{P}^2$ and therefore $h^1(\mathcal{O}_{B_1}) = h^1(\mathcal{O}_X) = h^1(\mathcal{O}_{\mathbb{P}^2}) = 0$, so that $B_1 \cong \mathbb{P}^1$. Then it follows from [S, Thm. A and Rmk. 1.6] that $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$ and π_1 is the first projection, π the second. Thus, we get that $\mathcal{F} \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}$ and $\mathcal{F}_1 \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 3}$. Then [Lo, Lemma 4.1] implies that $\mathcal{G} \cong \mathcal{O}_{\mathbb{P}^2}(-1)^{\oplus r}$ and $\mathcal{G}_1 \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$. But then $\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(2)^{\oplus r}) \cong \mathcal{E} \cong \pi^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus r})$, giving the contradiction

$$0 = (\pi_1^*(\mathcal{O}_{\mathbb{P}^1}(2r))^2 \cong c_1(\mathcal{E})^2 = \pi^*(\mathcal{O}_{\mathbb{P}^2}(r))^2 = r^2.$$

This proves that $(X, \mathcal{O}_X(1), \mathcal{E})$ is not a linear Ulrich triple over a smooth curve.

By Proposition 2 we get that $c_2(\mathcal{E}) \neq 0$ and, since $c_3(\mathcal{E}) = \pi^* c_3((\mathcal{G}(\det \mathcal{F})) = 0$, we find by Corollary 1 that $\mathcal{E}$ and then $\mathcal{G}$ have rank two. Let Z be an Ulrich subvariety arising from a general section of $\mathcal{E}$. It follows from Lemma 4.6 that $\mathcal{O}_Z \cong \pi^* \mathcal{O}_{Z_B}$, where Z_B is the zero locus of a general section of $\mathcal{G}(\det \mathcal{F})$. Since $\mathcal{G}(\det \mathcal{F})$ is globally generated (see for example [Li, Exc. 5.1.29(b)]) we have that Z_B is smooth and connected (since Z is) and therefore $Z_B = \{P\}$ for some point $P \in B$. By Lemma 4.5, there is a smooth irreducible curve $C \in |\det(\mathcal{G}(\det \mathcal{F}))|$ such that $P \in C$ and $\mathcal{O}_C(P)$ is globally generated, so that $C \cong \mathbb{P}^1$. Moreover, $H^1(\mathcal{O}_B(-C)) \cong H^1(\mathcal{O}_X(-\det \mathcal{E})) = 0$ by Lemma 4.2(ii), hence the exact sequence

$$0 \to \mathcal{O}_B(-C) \to \mathcal{O}_B \to \mathcal{O}_C \to 0$$

shows that $H^1(\mathcal{O}_B) = 0$. Now, the exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{G}(\det \mathcal{F}) \to \mathcal{J}_{\{P\}/B}(C) \to 0$$

gives, using Lemma 5.1(vii),

$$h^0(\mathcal{J}_{\{P\}/B}) = h^0(\mathcal{G}(\det \mathcal{F})) - 1 = h^0(\mathcal{E}) - 1 = 2d - 1.$$

Then, the exact sequence

$$0 \to \mathcal{O}_B \to \mathcal{J}_{\{P\}/B}(C) \to \mathcal{O}_{\mathbb{P}^1}(C^2 - 1) \to 0$$

shows, since $C^2 \geq 0$, that

$$c_1(\mathcal{G}(\det \mathcal{F}))^2 = C^2 = h^0(\mathcal{O}_{\mathbb{P}^1}(C^2 - 1)) = h^0(\mathcal{J}_{\{P\}/B}) - 1 = 2d - 2.$$

On the other hand, since $\mathcal{E}$ is semistable by [CH, Thm. 2.9], it satisfies Bogomolov's inequality $0 \leq (4c_2(\mathcal{E}) - c_1(\mathcal{E})^2)H^{n-2}$, that is

$$0 \leq \pi^*(4c_2(\mathcal{G}(\det \mathcal{F})) - c_1(\mathcal{G}(\det \mathcal{F}))^2)H^{n-2} = 4c_2(\mathcal{G}(\det \mathcal{F})) - c_1(\mathcal{G}(\det \mathcal{F}))^2 = 6 - 2d$$

that is $d \leq 3$. Since $\rho(X) \geq 2$ we deduce that $d = 3$ and $(X, \mathcal{O}_X(1)) \cong (\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1))$ (see for example [H2, Thm. 3.1]). Also, $c_1(\mathcal{E})^n = \pi^* c_1(\mathcal{G}(\det \mathcal{F}))^n = 0$ and we find that $\mathcal{E} = q^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2})$ by [LS, Cor. 4.9], where q is the second projection. This concludes the proof. $\square$

8. Surfaces

In the case of surfaces, Ulrich subvarieties are 0-dimensional and smooth, hence we need to understand when they can be a point.

Before giving a series of examples, we recall the following fact.

Remark 8.1. Let $\Gamma \subset \mathbb{P}^3$ be a smooth cubic surface. Then there are 72 classes of twisted cubics on Γ, listed in [CH, Ex. 3.5]. Note that, for any two twisted cubics T, T' with $T \not\sim T'$, we have that $T \cdot T' \geq 2$. In fact, assume that $m := T \cdot T' \leq 1$. Since $h^0(\mathcal{O}_\Gamma(T)) = 3$, the exact sequence

$$0 \to \mathcal{O}_\Gamma(T' - T) \to \mathcal{O}_\Gamma(T') \to \mathcal{O}_{\mathbb{P}^1}(m) \to 0$$

shows that $h^0(\mathcal{O}_\Gamma(T' - T)) \geq 3 - h^0(\mathcal{O}_{\mathbb{P}^1}(m)) \geq 1$. Hence $T' - T$ is effective and $H \cdot (T' - T) = 0$, implying that $T \sim T'$.

Example 8.2. Let $\Gamma \subset \mathbb{P}^3$ be a smooth cubic surface. All rank 2 Ulrich bundles $\mathcal{E}$ on Γ with $c_2(\mathcal{E}) = 1$ are of type $\mathcal{E} = \mathcal{O}_\Gamma(T)^{\oplus 2}$, where $T \subset \Gamma$ is a twisted cubic (as listed in [CH, Ex. 3.5]).

Indeed, it follows from [CH, Ex. 3.6] that $\mathcal{E}$ is an extension of type

$$0 \to \mathcal{O}_\Gamma(T) \to \mathcal{E} \to \mathcal{O}_\Gamma(T') \to 0$$

where T and T' are two twisted cubic curves contained in Γ. Then $1 = c_2(\mathcal{E}) = T \cdot T'$ and Remark 8.1 implies that $T \sim T'$. On the other hand $\operatorname{Ext}^1(\mathcal{O}_\Gamma(T'), \mathcal{O}_\Gamma(T)) = H^1(\mathcal{O}_\Gamma(T - T')) = H^1(\mathcal{O}_\Gamma) = 0$ and therefore the above sequence splits, hence $\mathcal{E} = \mathcal{O}_\Gamma(T)^{\oplus 2}$.

Next, we recall the notation for Hirzebruch surfaces $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ with $e \geq 0$, f a ruling and C_0 an irreducible curve with $C_0^2 = -e, C_0 \cdot f = 1$. In particular, a smooth non-degenerate cubic $S \subset \mathbb{P}^4$ is isomorphic to the Hirzebruch surface X_1 embedded with $H = C_0 + 2f$.

Example 8.3. Let $\Sigma \subset \mathbb{P}^4$ be a smooth non-degenerate cubic surface. Then $\mathcal{E} = \mathcal{O}_\Sigma(C_0 + f)^{\oplus 2}$ is a rank 2 Ulrich bundle with $c_2(\mathcal{E}) = 1$.

In fact, as said above, we have that $(\Sigma, H) \cong (X_1, C_0 + 2f)$. Now, $H^0(\mathcal{O}_{X_1}(C_0 + f - H)) = H^0(\mathcal{O}_{X_1}(-f)) = 0$ and, by Serre duality, $H^2(\mathcal{O}_{X_1}(C_0 + f - 2H)) = H^0(\mathcal{O}_{X_1}(-C_0))^* = 0$. Therefore $H^0(\mathcal{E}(-H)) = H^2(\mathcal{E}(-2H)) = 0$. Moreover $c_1(\mathcal{E}) = 2C_0 + 2f$ and $c_2(\mathcal{E}) = (C_0 + f)^2 = 1$ satisfy [C, (2.2)]. Hence $\mathcal{E}$ is Ulrich by [C, Prop. 2.2(4)].

As it turns out, the above examples are the only ones with connected Ulrich subvarieties, as we will see in the proof of Theorem 3. Before proving the theorem, we need two lemmas.

Lemma 8.4. *Let $S \subset \mathbb{P}^N$ be a smooth irreducible surface of degree $d \geq 2$ and let $\mathcal{E}$ be a rank $r \geq 2$ Ulrich bundle on S. Let Z be a nonempty Ulrich subvariety associated to $\mathcal{E}$. Then Z is connected if and only if $c_2(\mathcal{E}) = 1$. In the latter case, we have that $r = 2, c_1(\mathcal{E})^2 = 2d - 2$ and $2 \leq d \leq 3$.*

Proof. Since $\dim Z = 0$, Z is smooth by definition and $[Z] = c_2(\mathcal{E})$, we have that Z is connected if and only if $Z = \{P\}$ is a point, that is, if and only if $c_2(\mathcal{E}) = 1$. For the rest of the proof we assume that $c_2(\mathcal{E}) = 1$. Let $V \subset H^0(\mathcal{E})$ be a general subspace of dimension $r - 1$ and let $\varphi : V \otimes \mathcal{O}_S \to \mathcal{E}$. It follows from Remark 5.4 that $Z = D_{r-2}(\varphi)$ is an Ulrich subvariety associated to $\mathcal{E}$. In particular, $Z \neq \emptyset$ by Proposition 2 and since $[Z] = c_2(\mathcal{E})$ and Z is smooth, we have that $Z = \{P\}$ for some point $P \in S$. Now Theorem 1 implies that $r = 2$ and Lemma 4.5 gives a smooth irreducible curve $C \in |\det \mathcal{E}|$ with $P \in C$ and $\mathcal{O}_C(P)$ globally generated, so that $C \cong \mathbb{P}^1$. Note that $C^2 = c_1(\mathcal{E})^2 > 0$, for otherwise we have the contradiction $c_2(\mathcal{E}) = 0$ by Lemma 4.2(i). Therefore $H^1(\mathcal{O}_S(-C)) = 0$ by Kawamata-Viehweg's vanishing and the exact sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

implies that $H^1(\mathcal{O}_S) = 0$. From the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{J}_{\{P\}/S}(C) \to 0$$

we deduce, using Lemma 5.1(vii), that $h^0(\mathcal{J}_{\{P\}/S}(C)) = h^0(\mathcal{E}) - 1 = 2d - 1$. Therefore the exact sequence

$$0 \to \mathcal{O}_S \to \mathcal{J}_{\{P\}/S}(C) \to \mathcal{O}_{\mathbb{P}^1}(C^2 - 1) \to 0$$

gives that

$$c_1(\mathcal{E})^2 = C^2 = h^0(\mathcal{O}_{\mathbb{P}^1}(C^2 - 1)) = h^0(\mathcal{J}_{\{P\}/S}(C)) - 1 = 2d - 2.$$

On the other hand, since $\mathcal{E}$ is semistable by [CH, Thm. 2.9], it satisfies Bogomolov's inequality $4c_2(\mathcal{E}) - c_1(\mathcal{E})^2 \ge 0$, that is $0 < 2d - 2 = c_1(\mathcal{E})^2 \le 4$, so that $2 \le d \le 3$. □

Lemma 8.5. *Let $(S, H) = (X_1, C_0 + 2f)$ be a Hirzebruch surface. Then, the only rank 2 Ulrich bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 1$ is $\mathcal{E} = \mathcal{O}_{X_1}(C_0 + f)^{\oplus 2}$.*

Proof. We know by Example 8.3 that $\mathcal{E} = \mathcal{O}_{X_1}(C_0 + f)^{\oplus 2}$ is Ulrich on $(X_1, C_0 + 2f)$. Now, assume that $\mathcal{E}$ is a rank 2 Ulrich bundle $\mathcal{E}$ with $c_2(\mathcal{E}) = 1$ and let $\det \mathcal{E} = \mathcal{O}_X(D)$. Then $D = \alpha C_0 + \beta f$ and it follows from Lemma 8.4 that $D^2 = 4$, hence $\alpha \ge 1$ since D is nef by Lemma 5.1(iii) and $4 = D^2 = \alpha(2\beta - \alpha)$ implies that $\alpha = \beta = 2$. It follows from [A, Thm. 1.1(1)] that $\mathcal{E} = \mathcal{O}_{X_1}(C_0 + f)^{\oplus 2}$. □

We now prove Theorem 3.

Proof of Theorem 3. We know by Lemma 8.4 that Z is connected if and only if $c_2(\mathcal{E}) = 1$. In the cases (i)-(iii) we therefore have that Z is connected by Lemma 7.3(i) and Examples 8.2, 8.3. Vice versa, assume that Z is connected, so that $c_2(\mathcal{E}) = 1$ and $r = 2, 2 \le d \le 3$ by Lemma 8.4. If $d = 2$ we conclude by Lemma 7.3(i) that we are in case (i). If $d = 3$ we have that $N \le 4$. If $N = 3$ we get by Example 8.2 that we are in case (ii), while if $N = 4$ we know by Lemma 8.5 that we are in case (iii). □

9. Threefolds

In the case $\mathcal{E}$ is an Ulrich bundle on a threefold, we have a lot of information about connectedness of Ulrich subvarieties.

First, we can characterize precisely the non-big case.

Proposition 9.1. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible threefold and let $\mathcal{E}$ be a non-big Ulrich bundle on X with $c_2(\mathcal{E}) \ne 0$. Then:*

(i) *If $c_1(\mathcal{E})^3 > 0$, every Ulrich subvariety associated to $\mathcal{E}$ is connected.*
(ii) *If $c_1(\mathcal{E})^3 = 0$, the Ulrich subvarieties associated to $\mathcal{E}$ are connected if and only if $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^1 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^1}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), q^*(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}))$, where $q : \mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^2$ is the second projection.*

Proof. Let $\det \mathcal{E} = \mathcal{O}_X(D)$. If $c_1(\mathcal{E})^3 > 0$, it follows from [LM, Thm. 2] that either $\mathcal{E}$ is as in Lemma 7.7 and we are done, or $(X, \mathcal{O}_X(1), \mathcal{E}) = (Q, \mathcal{O}_Q(1), \mathcal{S})$ where $Q = Q_3$ and $\mathcal{S}$ is the spinor bundle. In the latter case, we have that $D = H$, hence $H^1(\mathcal{E}(-D)) = H^1(\mathcal{E}(-1)) = 0$, hence any Ulrich subvariety is connected by Lemma 7.6. This proves (i). In case (ii), we have by [LM, Thm. 2] and $c_2(\mathcal{E}) \ne 0$ that $(X, \mathcal{O}_X(1), \mathcal{E})$ is a linear Ulrich triple over a surface and we conclude by Lemma 7.9. □

In the big case, even though our results are not conclusive, they strongly restrict the possibilities, as follows.

Proposition 9.2. *Let $X \subset \mathbb{P}^N$ be a smooth irreducible threefold, let $\mathcal{E}$ be a big rank $r \ge 2$ Ulrich bundle on X and let Z be an Ulrich subvariety associated to $\mathcal{E}$. Then Z is nonempty and we have:*

(i) *If $\mathcal{E}$ is V-big and $r \ge 3$, then Z is connected.*
(ii) *If $\mathcal{E}$ is not V-big, then $(X, \mathcal{O}_X(1))$ is one of the following:*
  (a) *A linear $\mathbb{P}^2$-bundle over a smooth curve.*
  (b) *A del Pezzo 3-fold of degree d with $3 \le d \le 7$.*
  (c) *A quadric fibration over a smooth curve.*
  (d) *A linear $\mathbb{P}^1$-bundle over a smooth surface.*

Moreover, in case (b), then Z is connected if $3 \le d \le 5$ and if $r = 2$ and $6 \le d \le 7$.

Proof. Ulrich subvarieties associated to $\mathcal{E}$ are nonempty by Remark 5.6 and Lemma 5.1(viii). Also, $c_1(\mathcal{E})^3 > 0$ by [LM, Rmk. 2.2]. First, (i) follows from Theorem 1(iv). If $\mathcal{E}$ is not V-big, that is $\mathbf{B}_+(\mathcal{E}) = X$, then X is covered by lines by [Bu, Thm. 2] and it follows from [LP, Thm. 1.4] (for (d) use also [SV, Thm. 0.2]) that $(X, \mathcal{O}_X(1))$ is as in (a)-(d) above. In case (b), consider the classification of del Pezzo threefolds (see for example [LP, pages 860-861], [F, Table, page 710]). If $3 \le d \le 5$, we have that $\mathrm{Pic}(X) \cong \mathbb{Z}H$, hence Z is connected by Lemma 7.6(ii). To do the case $r = 2$ and $6 \le d \le 7$, we will use Lemma 5.1(vi). Suppose that $X \cong \mathbb{P}(T_{\mathbb{P}^2})$. Then Z is connected by [CFM1, Main Thm. for F] if $\mathcal{E}$ is indecomposable. If $\mathcal{E}$ is direct sum of two Ulrich line bundles, it is easy to show, using [CFM1, Cor. 2.7], that the only possibility is when $c_1(\mathcal{E}) = 2H$ and then Z is connected by Lemma 7.6(i). When

X is the blow-up of $\mathbb{P}^3$ at a point, it follows from [CFiM, Prop. 2.7 and Thm. A] that Z is connected. Finally, when $X \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, denote by H_i the pull-back of $\mathcal{O}_{\mathbb{P}^1}(1)$ by the i-th projection, $1 \le i \le 3$. We have that either $\mathcal{E}$ is indecomposable and Z is connected by [CFM2, Thm. B], unless we are in [CFM2, Thm. B, case (5)] or $\mathcal{E}$ is decomposable. Now, in case (5), $c_1(\mathcal{E})H^2 = 10$, while Lemma 5.1(x), gives that $c_1(\mathcal{E})H^2 = 12$, hence this case does not occur for Ulrich bundles. If $\mathcal{E}$ is decomposable, then it is direct sum of two Ulrich line bundles and we easily deduce, from [CFM2, Lemma 2.4], that each direct summand must be of type $2H_i + H_j$, for $i \ne j$. It follows that $H^1(-2H_i - H_j) = 0$ and hence $H^1(\mathcal{E}(-D)) \cong H^1(\mathcal{E}^*) = 0$. Hence Z is connected by Lemma 7.6(i). □

We remark that examples as in (a) with Z disconnected do exist, see Example 10.1. We do not know if there are others.

10. Some examples

We give some explicit examples, with various vanishings of Chern classes.

The following is an example of a big rank $r \ge 2$ Ulrich bundle $\mathcal{E}$ with $c_2(\mathcal{E}) \ne 0, c_2(\mathcal{E})^2 = c_3(\mathcal{E}) = 0$ and disconnected Ulrich subvarieties (even in rank 2).

Example 10.1. Let B be a smooth irreducible curve of genus g, let $\mathcal{F}$ be a rank $n \ge 2$ very ample bundle on B, let $X = \mathbb{P}(\mathcal{F})$ with bundle map $\pi : X \to B$ and let $H = \xi$ be the tautological line bundle on X. Note that $\xi^n \ge 2$, otherwise $X = \mathbb{P}^n$, a contradiction since $\rho(X) = 2$. Let M (respectively $\mathcal{G}$) be a line bundle (resp. a rank $r-1$ vector bundle, with $r \ge 2$) on B such that $H^i(M) = H^i(\mathcal{G}) = 0$ for $i \ge 0$. Let $\mathcal{L} = \xi + \pi^* M$. Then $\mathcal{L}$ is an Ulrich line bundle for $(X, \mathcal{O}_X(1))$: In fact $\mathcal{L}(-pH) = (1-p)\xi + \pi^* M$. Since $R^j\pi_*((1-p)\xi) = 0$ for $j \ge 1, 1 \le p \le n$ and for $j = 0, p \ge 2$, we get, for $i \ge 0$, that $H^i(\mathcal{L}(-pH)) = 0$ for $2 \le p \le n$ and $H^i(\mathcal{L}(-H)) = H^i(M) = 0$. Let $\mathcal{E}$ be any bundle sitting in an extension of type

$$0 \to \mathcal{L} \to \mathcal{E} \to \pi^*(\mathcal{G}(\det \mathcal{F})) \to 0.$$

It follows from [Lo, Lemma 4.1] that $\mathcal{E}$ is an Ulrich bundle for $(X, \mathcal{O}_X(1))$. Setting

$$N = c_1(\mathcal{G}(\det \mathcal{F})) = c_1(\mathcal{G}) + (r-1)c_1(\mathcal{F})$$

we have

$$c_1(\mathcal{E}) = \mathcal{L} + \pi^* N, c_2(\mathcal{E}) = c_1(\mathcal{L})c_1(\pi^*(\mathcal{G}(\det \mathcal{F})) = (\xi + \pi^* M)\pi^* N = \xi\pi^* N.$$

Since $\mathcal{L}$ and $\pi^*(\mathcal{G}(\det \mathcal{F}))$ are Ulrich, we have that $\mathcal{L}$ and $\pi^* N$ are globally generated. Moreover, $\pi^* N$ is not trivial by Lemma 5.1(iii). Therefore $c_2(\mathcal{E}) = \xi\pi^* N \ne 0$. Also, we have that $\xi^n + \xi^{n-1}\pi^* M > 0$: In fact, if $g \ge 1$, we have that $\pi^* M$ is nef, hence $\xi^n + \xi^{n-1}\pi^* M \ge \xi^n > 0$. If $g = 0$, we have that $M = \mathcal{O}_{\mathbb{P}^1}(-1)$, hence $\xi^n + \xi^{n-1}\pi^* M = \xi^n - 1 > 0$. Therefore

$$c_1(\mathcal{E})^n = \mathcal{L}^n + \mathcal{L}^{n-1}\pi^* N = \xi^n + \xi^{n-1}\pi^* M + \mathcal{L}^{n-1}\pi^* N \ge \xi^n + \xi^{n-1}\pi^* M > 0. \tag{10.1}$$

Moreover $c_2(\mathcal{E})^2 = \xi^2\pi^* N^2 = 0$, and, for $i \ge 3$,

$$c_i(\mathcal{E}) = \sum_{j=0}^{i} c_j(\mathcal{L})c_{i-j}(\pi^*(\mathcal{G}(\det \mathcal{F})) = c_i(\pi^*(\mathcal{G}(\det \mathcal{F})) + c_{i-1}(\pi^*(\mathcal{G}(\det \mathcal{F}))c_1(\mathcal{L}) =$$
$$= \pi^* c_i(\mathcal{G}(\det \mathcal{F})) + \pi^* c_{i-1}(\mathcal{G}(\det \mathcal{F}))c_1(\mathcal{L}) = 0.$$

Now, assume that $n \ge 3$ and let Z be any Ulrich subvariety associated to $\mathcal{E}$. Then Z is nonempty by Proposition 2. Let $D = \mathcal{L} + \pi^* N$ and consider the exact sequence

$$0 \to \mathcal{O}_X(-D)^{\oplus(r-1)} \to \mathcal{E}(-D) \to \mathcal{J}_{Z/X} \to 0.$$

Since D is big by (10.1) and nef by Lemma 5.1(iii), we have that $H^1(\mathcal{O}_X(-D)) = H^2(\mathcal{O}_X(-D)) = 0$ by Kawamata-Viehweg. Thus,

$$h^1(\mathcal{J}_{Z/X}) = h^1(\mathcal{E}(-D)) = h^1(\pi^*(\mathcal{G}(\det \mathcal{F}))(-D)) + h^1(\mathcal{L}(-D)).$$

Now, $h^1(\pi^*(\mathcal{G}(\det \mathcal{F}))(-D)) = h^1(\pi^*(\mathcal{G}(\det \mathcal{F} - N - M))(-\xi)) = 0$, since $R^j\pi_*(-\xi) = 0$ for $j \ge 0$. Also, $0 = \chi(\mathcal{G}) = \deg \mathcal{G} + (r-1)(1-g)$, hence $\deg \mathcal{G} = (r-1)(g-1)$ and $\deg N = (r-1)(\deg \mathcal{F} + g - 1) > 0$, since $\mathcal{F}$ is very ample. Therefore $h^0(-N) = 0$ and $h^1(-N) = \deg N + g - 1 = (r-1)\deg \mathcal{F} + r(g-1)$. It follows that

$$h^1(\mathcal{L}(-D)) = h^1(\pi^*(-N)) = h^1(-N) = (r-1)\deg \mathcal{F} + r(g-1)$$

so that

$$h^1(\mathcal{J}_{Z/X}) = (r-1)\deg \mathcal{F} + r(g-1).$$

Now the exact sequence

$$0 \to \mathcal{J}_{Z/X} \to \mathcal{O}_X \to \mathcal{O}_Z \to 0$$

and $h^1(\mathcal{O}_X) = h^1(\mathcal{O}_B) = g$ show that

$$h^0(\mathcal{O}_Z) \geq h^1(\mathcal{J}_{Z/X}) + 1 - g = (r-1)(\deg \mathcal{F} + g - 1) \geq 2$$

hence Z is disconnected. Finally we show that $\mathcal{E}$ is big. In fact, using Lemma 4.4, we have

$$s_n(\mathcal{E}^*) = [\xi + \pi^*(M+N)]^n - (n-1)[\xi + \pi^*(M+N)]^{n-2}\xi\pi^*N = \xi^n + \xi^{n-1}(\pi^*N + n\pi^*M) =$$

$$= \deg \mathcal{F} + \deg N + n \deg M = r \deg \mathcal{F} + \deg \mathcal{G} + n \deg M = r \deg \mathcal{F} + (r+n-1)(g-1) > 0$$

both if $g \geq 1$ and if $g = 0$ since in this case $\deg \mathcal{F} \geq n$. Thus $\mathcal{E}$ is big but not V-big if $r \geq 3$ by Theorem 1(iv).

For an explicit instance of the above, let $X = \mathbb{P}^1 \times \mathbb{P}^{n-1} \subset \mathbb{P}^N$ embedded with the Segre embedding and let $\mathcal{E} = \mathcal{O}_X(n-1,0)^{\oplus(r-1)} \oplus \mathcal{O}_X(0,1)$.

The following is an example of a rank $r \in \{2,3\}$ globally generated bundle $\mathcal{E}$ with $c_3(\mathcal{E}) = 0, c_2(\mathcal{E})^2 \neq 0, H^0(\mathcal{E}^*) = 0$ and disconnected degeneracy locus $D_{r-2}(\varphi)$.

Example 10.2. Let $Q \subset \mathbb{P}^4$ be a smooth quadric, let $W = \mathbb{P}^2 \times Q$ and consider the Segre embedding $W \subset \mathbb{P}^{14}$ given by $L = \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_Q(1)$. Note that $K_W = -3L$, so that W is a Mukai 5-fold that is scheme-theoretically cut out by quadrics in $\mathbb{P}^{14}$ (see for example [Ru, Rmk. 26]. Let $X \in |L|$ be a general hyperplane section of W and let $H = L_{|X}$. It follows from [BS, Ex. 14.1.5] that X contains exactly two fibers, that are a linear $\mathbb{P}^2$ in $\mathbb{P}^{14}$, say $F_i = \mathbb{P}^2 \times \{z_i\}, i = 1,2$, with $N_{F_i/X} \cong \Omega_{\mathbb{P}^2}(1)$. Note that $W \cong \mathbb{P}(\mathcal{O}_Q(1)^{\oplus 3})$ and $X = Z(\sigma)$, with $\sigma \in H^0(\mathcal{O}_Q(1)^{\oplus 3})$ a general section. In particular we can assume that z_1, z_2 are general points of Q. Let $Z = F_1 \sqcup F_2 \subset X$. We first observe that $\mathcal{J}_{Z/X}(H)$ is globally generated. In fact, we have a surjection $\mathcal{J}_{Z/W}(L) \to \mathcal{J}_{Z/X}(H)$, hence it is enough to prove that $\mathcal{J}_{Z/W}(L)$ is globally generated. On the other hand, we have that the linear span $\langle Z \rangle$ is a linear $\mathbb{P}^5$ in $\mathbb{P}^{14}$ and of course $\mathcal{J}_{\langle Z \rangle/\mathbb{P}^{14}}(1)$ is globally generated. Since we have a surjection $\mathcal{J}_{\langle Z \rangle/\mathbb{P}^{14}}(1) \to \mathcal{J}_{\langle Z \rangle \cap W/W}(L)$, we just need to prove that $\langle Z \rangle \cap W = Z$. To this end, let $w \in \langle Z \rangle \cap W$. Then $w \in \langle f_1, f_2 \rangle$, with $f_1, f_2 \in Z$. If $f_1, f_2 \in F_1$ (or in F_2), then $w \in \langle f_1, f_2 \rangle \subset F_1 \subset Z$. If $f_1 \in F_1$ and $f_2 \in F_2$, then we observe that the line $\langle f_1, f_2 \rangle$ is not contained in W. In fact, it is well-known that the only lines in W are of type $R \times \{p\}$, with $R \subset \mathbb{P}^2$ a line, or $\{p\} \times R$, with $R \subset Q$ a line. Both cases are excluded since $f_i = (x_i, z_i)$ with $x_1 \neq x_2, z_1 \neq z_2$. Therefore $\langle f_1, f_2 \rangle \not\subset W$ and since W is scheme-theoretically cut out by quadrics, we can find a quadric $Q' \subset \mathbb{P}^{14}$ such that $W \subset Q'$ and $\langle f_1, f_2 \rangle \not\subset Q'$. But then $w \in \langle f_1, f_2 \rangle \cap W \subset \langle f_1, f_2 \rangle \cap Q' = \{f_1, f_2\} \subset Z$. Thus, we have proved that $\mathcal{J}_{Z/X}(H)$ is globally generated. Now we have

$$(\Lambda^2 N_{Z/X})(-H_{|Z}) = \mathcal{O}_Z(K_X - K_Z - H) \cong \mathcal{O}_Z(-3H - K_Z) \cong \mathcal{O}_Z$$

since, on each $F_i \cong \mathbb{P}^2$ we have that $\mathcal{O}_{F_i} \otimes \mathcal{O}_Z(-3H - K_Z) \cong \mathcal{O}_{\mathbb{P}^2}$. Moreover, $H^2(\mathcal{O}_X(-H)) = 0$ by Kodaira vanishing and therefore the Hartshorne-Serre correspondence gives a rank two vector bundle $\mathcal{F}$ on X, with $\det \mathcal{F} = H$, sitting in an exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{F} \to \mathcal{J}_{Z/X}(H) \to 0. \tag{10.2}$$

Also, note that $H^1(\mathcal{O}_X) = 0$ by the exact sequence

$$0 \to \mathcal{O}_W(-L) \to \mathcal{O}_W \to \mathcal{O}_X \to 0$$

and the facts that $H^1(\mathcal{O}_W) = 0$ by Künneth, since $H^1(\mathcal{O}_{\mathbb{P}^2}) = H^1(\mathcal{O}_Q) = 0$ and $H^2(\mathcal{O}_W(-L)) = 0$ by Kodaira vanishing. Hence $\mathcal{F}$ is globally generated. Moreover $\mathcal{F}^* \cong \mathcal{F}(-H)$ and twisting (10.2) we get

$$0 \to \mathcal{O}_X(-H) \to \mathcal{F}^* \to \mathcal{J}_{Z/X} \to 0$$

and Kodaira vanishing gives $H^0(\mathcal{F}^*) = 0, h^1(\mathcal{F}^*) = h^1(\mathcal{J}_{Z/X}) = h^0(\mathcal{O}_Z) - h^0(\mathcal{O}_X) = 1$. Also, we have $c_2(\mathcal{F}) = [Z] = [F_1] + [F_2]$ and therefore $c_2(\mathcal{F})^2 = [F_1]^2 + [F_2]^2 = 2$, since the self-intersection formula

[H1, Appendix A, C.7] shows that $[F_i]^2 = c_2(N_{F_i/X}) = c_2(\Omega_{\mathbb{P}^2}(1)) = 1$. Finally, if $r = 2$ we set $\mathcal{E} = \mathcal{F}$. If $r = 3$, let $0 \neq e \in H^1(\mathcal{F}^*)$ and consider the associated extension

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{F} \to 0.$$

Since $\mathcal{O}_X$ and $\mathcal{F}$ are locally free, we have that so is $\mathcal{E}$: In fact $\mathcal{E}xt^i_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) = 0$ for $i > 0$ because the same holds for $\mathcal{O}_X$ and $\mathcal{F}$. Moreover, $\mathcal{E}$ is globally generated since $H^1(\mathcal{O}_X) = 0$ and $c_3(\mathcal{E}) = c_3(\mathcal{F}) = 0, c_2(\mathcal{E})^2 = c_2(\mathcal{F})^2 = 2$. Moreover, dualizing the above exact sequence we get

$$0 \to \mathcal{F}^* \to \mathcal{E}^* \to \mathcal{O}_X \to 0$$

with cohomology sequence, using $H^0(\mathcal{F}^*) = 0, H^1(\mathcal{F}^*) = \mathbb{C}e$,

$$0 \to H^0(\mathcal{E}^*) \to H^0(\mathcal{O}_X) \xrightarrow{\delta} H^1(\mathcal{F}^*)$$

where, by construction, $\delta(1) = e$, so that δ is an isomorphism and $H^0(\mathcal{E}^*) = 0$. The fact that $D_{r-2}(\varphi)$ is disconnected for any $\varphi : \mathcal{O}_X^{\oplus(r-1)} \to \mathcal{E}$ such that $D_{r-2}(\varphi)$ is reduced of pure codimension 2, follows from Theorem 1. As a matter of fact, one of these degeneracy loci is just Z.

The following is an example of a rank 4 Ulrich bundle $\mathcal{E}$ with $c_3(\mathcal{E}) \neq 0, c_2(\mathcal{E})^2 \neq 0, c_4(\mathcal{E}) = 0, H^0(\mathcal{E}^*) = 0$ and disconnected degeneracy locus $D_{r-3}(\varphi)$.

Example 10.3. Let $Q \subset \mathbb{P}^5$ be a smooth quadric and let $\mathcal{E} = \mathcal{S}' \oplus \mathcal{S}''$. It is easily checked, using [O1, Rmk. 2.9], that $c_2(\mathcal{E}) = 2(e_2 + e_2')$, where e_2, e_2' generate $H^4(Q, \mathbb{Z})$. Also, $c_3(\mathcal{E}) = 2H^3, c_4(\mathcal{E}) = 0$ and $c_2(\mathcal{E})^2 = 8$. Now, for any $\varphi : \mathcal{O}_X^{\oplus 3} \to \mathcal{E}$ such that $D_1(\varphi)$ is reduced of pure codimension 3, we have that $D_1(\varphi)$ has exactly two connected components: it has at least two by Theorem 1 and no more since $[D_1(\varphi)] = c_3(\mathcal{E}) = 2H^3$.

11. Results in rank 2

We can give a precise result when $H^1(\mathcal{O}_X) = 0$. In particular, the next lemma shows that, in rank 2, connectedness of degeneracy loci (or of Ulrich subvarieties), in the case $k = 2$, is not governed by Chern classes as in the case of higher rank.

Lemma 11.1. *Let X be a smooth variety with $H^1(\mathcal{O}_X) = 0$. Let $\mathcal{E}$ be a globally generated vector bundle of rank 2 on X with $c_2(\mathcal{E}) \neq 0$ and $H^0(\mathcal{E}^*) = 0$ (which holds, in particular, if $\mathcal{E}$ is Ulrich). Let $h = h^1(\mathcal{E}^*)$ and let $\sigma \in H^0(\mathcal{E})$ be a general section. Then we have:*

(i) *$Z(\sigma)$ has $h+1$ connected components. In particular $Z(\sigma)$ is connected if and only if $H^1(\mathcal{E}^*) = 0$.*
(ii) *If $H^1(\mathcal{E}^*) \neq 0$, there is a globally generated bundle $\tilde{\mathcal{E}}$ of rank $h+2$ on X with $H^0(\tilde{\mathcal{E}}^*) = H^1(\tilde{\mathcal{E}}^*) = 0$, such that $\mathcal{E}$ arises as a quotient, with $Z(\sigma) = D_h(\psi)$,*

$$0 \to \mathcal{O}_X^{\oplus h} \xrightarrow{\psi} \tilde{\mathcal{E}} \to \mathcal{E} \to 0.$$

Vice versa, given a globally generated bundle $\tilde{\mathcal{E}}$ as in (ii), then $Z(\sigma)$ has $h+1$ connected components.

Proof. Note that $Z(\sigma) \neq \emptyset$ by Lemma 4.1 and $h = h^1(\mathcal{E}^*) = h^1(\mathcal{J}_{Z(\sigma)/X})$ by Lemma 6.5(c). On the other hand, as $H^1(\mathcal{O}_X) = 0$, we have the exact sequence

$$0 \to H^0(\mathcal{O}_X) \to H^0(\mathcal{O}_{Z(\sigma)}) \to H^1(\mathcal{J}_{Z(\sigma)/X}) \to 0$$

showing that $h^0(\mathcal{O}_{Z(\sigma)}) = h + 1$, that is (i). Now (ii) follows from Lemma 6.5(b). Vice versa, given a globally generated bundle $\tilde{\mathcal{E}}$ as in (ii), then $Z(\sigma)$ has $h+1$ connected components by Lemma 6.5(a). □

We also have the following.

Lemma 11.2. *Let $\mathcal{E}$ be a rank 2 globally generated bundle on $X \subset \mathbb{P}^N$ with $c_2(\mathcal{E}) \neq 0$ and $n \geq 3$. Let $\sigma \in H^0(\mathcal{E})$ be a general section and set $Z = Z(\sigma)$. Then:*

(i) *If $c_1(\mathcal{E})^3 \neq 0$ and $H^1(\mathcal{E}^*) = 0$, then Z is connected.*
(ii) *Let Y be as in Lemma 4.5 and assume that Y is smooth (which holds, for example, if $n \leq 3$ or if $c_2(\mathcal{E})^2 = 0$ by Porteous's formula), $h^0(\mathcal{O}_Y(Z)) = 2$, $c_1(\mathcal{E})^3 \neq 0, H^0(\mathcal{E}^*) = 0$ and $H^1(\mathcal{O}_X) = 0$. Then Z is connected.*
(iii) *If $n \geq 4$, $\mathcal{E}$ is Ulrich and $c_1(\mathcal{E})^3 = 0$, then Z is disconnected.*

(iv) *If* $n \geq 4, c_3(\mathcal{E}) = 0$, $\mathcal{E}$ *is Ulrich and* X *is subcanonical, then* Z *is connected unless* $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), \pi^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus 2})$*, where* $\pi : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ *is a projection, and, in the latter case,* Z *has exactly* 4 *connected components.*

Proof. If we set $\det \mathcal{E} = \mathcal{O}_X(D)$, then, since $\mathcal{E}$ has rank 2, we have that $\mathcal{E}(-D) \cong \mathcal{E}^*$. Now (i) follows from Lemma 7.6(i). Under hypothesis (ii), the exact sequence (4.2) becomes

$$0 \to \mathcal{E}^* \to \mathcal{O}_X^{\oplus 2} \to \mathcal{O}_Y(Z) \to 0$$

and it follows that $H^1(\mathcal{E}^*) = 0$, so that Z is connected by (i). This proves (ii). (iii) is just Remark 7.8. Finally, (iv) follows from Remark 6.10. □

Note that (iii) and (iv) above are not in conflict, even though if $c_1(\mathcal{E})^3 = 0$, then $c_3(\mathcal{E}) = 0$ by Lemma 4.2(i). In fact, as proved in Remark 6.10, if $c_1(\mathcal{E})^3 = 0$ and X is subcanonical, then $(X, \mathcal{O}_X(1), \mathcal{E}) = (\mathbb{P}^2 \times \mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1), \pi^*(\mathcal{O}_{\mathbb{P}^2}(2))^{\oplus 2})$.

On the other hand, when $c_1(\mathcal{E})^3 \neq 0$, we have that Z can be both connected (for example on a linear Ulrich triple over a threefold) or disconnected (even when $c_1(\mathcal{E})^n > 0$), as in Example 10.1.

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Valerio Buttinelli, Dipartimento di Matematica "Guido Castelnuovo", Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Roma, Italy. email: valerio.buttinelli@uniroma1.it

Angelo Felice Lopez, Dipartimento di Matematica e Fisica, Università di Roma Tre, Largo San Leonardo Murialdo 1, 00146, Roma, Italy. e-mail angelo.lopez@uniroma3.it

Roberto Vacca, Dipartimento di Matematica, Università di Roma Tor Vergata, Via della Ricerca Scientifica, 00133 Roma, Italy. email: vacca@mat.uniroma2.it