A LINEAR LOWER BOUND ON THE ULRICH COMPLEXITY OF HYPERSURFACES

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Abstract. We give a lower bound on the Ulrich complexity of hypersurfaces in terms of their dimension.

1. Introduction

Let $X \subset \mathbb{P}^N$ be a smooth irreducible variety of dimension n. In order to study the geometry of X, in recent years, an interesting point of view has emerged, the one of Ulrich bundles. An Ulrich bundle is a vector bundle $\mathcal{E}$ on X such that $H^i(\mathcal{E}(-p)) = 0$ for all $i \ge 0$ and $1 \le p \le n$. In the presence of such bundles, several geometrical features can be discovered on X, see for example the seminal paper [ES], the survey [B] and the book [CMRPL]. The main open problem about Ulrich bundles is their existence, conjectured to happen in all cases. Aside for curves and several surfaces, there are a few families of varieties that are known to carry an Ulrich bundle, among which complete intersections, by [HUB]. Once the existence is established, one defines the Ulrich complexity as

$$\mathrm{uc}(X, \mathcal{O}_X(1)) = \min\{r \ge 1 : \text{ there exists a rank } r \text{ Ulrich bundle on } X\}.$$

We simply write $\mathrm{uc}(X)$ when $\mathcal{O}_X(1)$ is naturally given.

In the case of a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $d \ge 2$ (when $d = 1$, $\mathrm{uc}(X) = 1$), Ulrich complexity of X falls within the Buchweitz, Greuel and Schreyer's conjecture [BGS], a stronger conjecture regarding aCM bundles, implying that $\mathrm{uc}(X) \ge 2^{\lfloor \frac{n-1}{2} \rfloor}$ (sharp for $d = 2$). It is also conjectured in [RT1] that $\mathrm{uc}(X) \ge 2^{\lfloor \frac{n+1}{2} \rfloor}$ when X is general. The above conjectures are wide open, the only known general result, of a slightly different flavour, being [E]. As far as we know, the best-known lower bound in terms of the dimension was shown in [BES, Thm. 3.1]:

$$\mathrm{uc}(X) \ge \sqrt{n+2} - 1.$$

It is the purpose of this paper to improve the above bound to a lower bound that is essentially the dimension. In order to state it precisely, let us define, for integers $m \ge 3, d \ge 3$, the functions

$$B(m) = \frac{30\pi^2(2m-1)(2m-3)}{24d^2(2m-3)\sin^2(\frac{\pi}{d}) + 5\pi d(2m-1)\sin(\frac{\pi}{d})}$$

and

$$F(m) = \min\{2m, B(m)\}.$$

Observe that $F(m) = 2m$ for $m \ge 21$, see Section 5. It is also possible that, for $3 \le m \le 20$, still $F(m) = 2m$. The precise values of d for which the latter holds are given in Table 1. For the other values of d, note that $F(m) \ge \frac{3m}{2}$ for $3 \le m \le 20$. Then we have

Theorem 1.
Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension $n \ge 6$ and degree $d \ge 3$. If n is odd we have

$$\mathrm{uc}(X) \ge \begin{cases} n-1 & \text{if } n \ge 43 \\ F(\frac{n-1}{2}) & \text{if } 7 \le n \le 41 \end{cases}$$

while if n is even we have

$$\mathrm{uc}(X) \ge \begin{cases} n-2 & \text{if } n \ge 44 \\ F(\frac{n-2}{2}) & \text{if } 6 \le n \le 42. \end{cases}$$

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Moreover, if n is even and X is very general, then

$$\mathrm{uc}(X) \geq \begin{cases} n & \text{if } n \geq 44 \\ F(\frac{n}{2}) & \text{if } n \leq 42. \end{cases}.$$

Similar calculations can be performed for $n \in \{4,5\}$, but they give $\mathrm{uc}(X) \geq 4$, a result that is already known (see for example [LR1, RT1, RT2] where several lower bounds are given).

Note that $F(m) = 2m$ for any $m \geq 3$ when $d = 3$ by Table 1. Hence, if X is a smooth cubic hypersurface of dimension $n \geq 6$, then $\mathrm{uc}(X) \geq n-1$ if n is odd, $\mathrm{uc}(X) \geq n-2$ if n is even and $\mathrm{uc}(X) \geq n$ if n is even and X is very general. On the other hand, since r is divisible by 3 for any rank r Ulrich bundle on X [KS, Prop. 2.5], [FK], the above lower bounds improve to the first multiple of 3.

We work over the complex numbers.

2. Homogeneous symmetric polynomials and series

In order to study the Chern character of an Ulrich bundle, we will need a few elementary estimates.

Definition 2.1. For each $n \geq 1, N \geq 1$ we denote by

$$h_n(X_1, \dots, X_N) = \sum_{\substack{i_1+\ldots+i_N=n \\ i_1\geq 0 \ldots i_N \geq 0}} X_1^{i_1} \cdots X_N^{i_N}$$

the *complete homogeneous symmetric polynomial of degree n in the variables $X_1, \dots, X_N$*. We set $h_0 = 1$.

Then we have

Lemma 2.2. *For $i \geq 1$, let $x_i \in \mathbb{C}$ be such that $\sum\limits_{i\geq 1} x_i$ is absolutely convergent and let $M > 0$ be such that $|x_i| \leq M$ for all $i \geq 1$. Let*

$$G(z) = \prod_{i \geq 1} (1 - x_i z)^{-1}.$$

Then $G(z)$ is well-defined for $|z| \leq R$ and any R with $0 < R < \frac{1}{M}$. Moreover

$$G(z) = \sum_{n \geq 0} h_n(x_1, x_2, \dots) z^n \tag{2.1}$$

where $h_n(x_1, x_2, \dots) = \sum\limits_{\substack{m_1+m_2+\ldots=n \\ m_1\geq 0, m_2 \geq 0, \ldots}} \prod\limits_{i \geq 1} x_i^{m_i}$ is also absolutely convergent.

Proof. Since $\sum_{i\geq 1} x_i$ is absolutely convergent, we have that $|x_i| \to 0$ for $i \to \infty$, hence an $M > 0$ such that $|x_i| \leq M$ for all $i \geq 1$ exists. If $|z| \leq R$, we have that $|1 - x_i z| \geq 1 - |x_i||z| \geq 1 - MR > 0$ and therefore

$$\left| \frac{1}{1 - x_i z} - 1 \right| = \frac{|x_i||z|}{|1 - x_i z|} \leq \frac{|x_i| R}{1 - MR}. \tag{2.2}$$

Since $(1 - x_i z)^{-1} \neq 0$ for all $i \geq 1$, it follows by [A, Chapt. 5, Thm. 6] that the product $G(z) = \prod_{i\geq 1}(1 - x_i z)^{-1}$ is absolutely convergent, hence well-defined. Also, we get that

$$G(z) = \lim_{N \to \infty} G_N(z)$$

where

$$G_N(z) = \prod_{i \geq 1}^{N} (1 - x_i z)^{-1} = \sum_{n \geq 0} h_n(x_1, \dots, x_N) z^n.$$

Moreover, the Weierstrass M-test and (2.2) imply that $G_N(z)$ converges uniformly to $G(z)$ for $|z| \leq R$. Now note that the partial sums of the series $h_n(x_1, x_2, \dots)$ are exactly the $h_n(x_1, \dots, x_N)$ and

$$|h_n(x_1, \dots, x_N)| \leq \sum_{\substack{i_1+\ldots+i_N=n \\ i_1\geq 0 \ldots i_N \geq 0}} |x_1|^{i_1} \cdots |x_N|^{i_N} \leq \left(\sum_{i=1}^{N} |x_i| \right)^n.$$

Since all sums $\sum_{i=1}^{N} |x_i|$ are bounded by hypothesis, it follows that $h_n(x_1, x_2, \ldots)$ is absolutely convergent and

$$\lim_{N\to\infty} h_n(x_1, \ldots, x_N) = h_n(x_1, x_2, \ldots).$$

Finally, Cauchy's formula gives that the coefficient of z^n in the Maclaurin series of $G(z)$ is precisely the limit, for $N \to \infty$, of the coefficients of z^n in the Maclaurin series of $G_N(z)$ and this proves (2.1). □

We now consider the function that will calculate the Chern character of an Ulrich bundle.

Lemma 2.3. *Let $d \in \mathbb{Z}, d \geq 3$, let $u = \frac{d-1}{2}$, let*

$$f(z) = \frac{d}{\sum_{k=0}^{d-1} e^{(u-k)z}} = \frac{d}{e^{uz} + e^{(u-1)z} + \ldots + e^{-uz}}$$

and consider its Maclaurin series

$$f(z) = 1 + \sum_{j\geq 1} (-1)^j b_j \frac{z^{2j}}{(2j)!}. \tag{2.3}$$

If $q := \frac{d^2}{4\pi^2}$ we have, for all $j \geq 1$, that the following estimates hold:

$$\frac{5}{4} q^j \leq \frac{b_j}{(2j)!} \leq \frac{2\sin(\frac{\pi}{d})}{\frac{\pi}{d}} q^j. \tag{2.4}$$

Proof. Note that $f(z)$ is holomorphic in a neighborhood of 0 and $f(0) = 1$. Moreover $f(z)$ is an even function, hence (2.3) makes sense. By the identity

$$e^{uz} + e^{(u-1)z} + \ldots + e^{-uz} = \sum_{k=0}^{d-1} e^{(u-k)z} = \frac{\sinh(\frac{dz}{2})}{\sinh(\frac{z}{2})}$$

we get that

$$f(z) = \frac{d\sinh(\frac{z}{2})}{\sinh(\frac{dz}{2})}. \tag{2.5}$$

By Euler's product formula (that can be obtained from [A, (24), Chapt. 5, §2.3] by replacing $\sinh(z) = -i\sin(iz)$),

$$\sinh(z) = z \prod_{\ell\geq 1} \left(1 + \frac{z^2}{\pi^2 \ell^2}\right)$$

we get from (2.5), that

$$f(z) = \frac{d\sinh(\frac{z}{2})}{\sinh(\frac{dz}{2})} = \frac{\frac{dz}{2} \prod_{\ell\geq 1} \left(1 + \frac{z^2}{4\pi^2\ell^2}\right)}{\frac{dz}{2} \prod_{\ell\geq 1} \left(1 + \frac{d^2z^2}{4\pi^2\ell^2}\right)} = \prod_{\substack{\ell\geq 1 \\ d\nmid \ell}} \left(1 + \frac{d^2z^2}{4\pi^2\ell^2}\right)^{-1}. \tag{2.6}$$

Now, it follows by Lemma 2.2 that

$$\frac{b_j}{(2j)!} = q^j h_j \tag{2.7}$$

where $h_j := h_j(\frac{1}{\ell^2}, d \nmid \ell, \ell \geq 1, \ldots)$ is the series of degree j in the positive variables $\frac{1}{\ell^2}, \ell \geq 1, d \nmid \ell$.

For the lower bound in (2.4), just observe that $h_j \geq 1 + \frac{1}{4} = \frac{5}{4}$, which gives, using (2.7), that $\frac{b_j}{(2j)!} \geq \frac{5}{4} q^j$.

For the upper bound in (2.4), using [A, Chapt. 5, Ex.1, §2.2 and (24), §2.3], we have

$$\frac{1}{2} = \prod_{\ell\geq 2} \left(1 - \frac{1}{\ell^2}\right) = \prod_{\substack{\ell\geq 2 \\ d\nmid \ell}} \left(1 - \frac{1}{\ell^2}\right) \prod_{\ell'\geq 1} \left(1 - \frac{1}{d^2(\ell')^2}\right) = \prod_{\substack{\ell\geq 2 \\ d\nmid \ell}} \left(1 - \frac{1}{\ell^2}\right) \frac{\sin(\frac{\pi}{d})}{\frac{\pi}{d}}$$

hence

$$\prod_{\substack{\ell\ge 2\\ d\nmid \ell}}\left(1-\frac{1}{\ell^2}\right)=\frac{\pi}{2d\sin(\frac{\pi}{d})}. \tag{2.8}$$

Set now $x_\ell=\frac{1}{\ell^2}$ for $\ell\ge 2$ with $d\nmid\ell$ in Lemma 2.2. Since $\frac{1}{\ell^2}\le\frac14$, we can set $M=\frac14$ in Lemma 2.2 and therefore the convergence holds, for example, for $|z|\le 3$. Then (2.1) gives

$$\prod_{\substack{\ell\ge 2\\ d\nmid \ell}}\left(1-\frac{1}{\ell^2}\right)^{-1}=\prod_{\substack{\ell\ge 2\\ d\nmid \ell}}(1-x_\ell)^{-1}=G(1)=\sum_{n\ge 0}h_n(\frac{1}{\ell^2},d\nmid\ell,\ell\ge 2,\dots). \tag{2.9}$$

On the other hand, using (2.8) and (2.9), we see that

$$h_j\le\sum_{n\ge 0}h_n(\frac{1}{\ell^2},d\nmid\ell,\ell\ge 2,\dots)=\prod_{\substack{\ell\ge 2\\ d\nmid \ell}}\left(1-\frac{1}{\ell^2}\right)^{-1}=\frac{2d\sin(\frac{\pi}{d})}{\pi}.$$

Therefore (2.7) gives that

$$\frac{b_j}{(2j)!}=q^jh_j\le\frac{2\sin(\frac{\pi}{d})}{\frac{\pi}{d}}q^j$$

and the lemma is proved. □

3. $\mathbb{Q}$-twisted vector bundles and their Chern character

We recall the notion of $\mathbb{Q}$-twisted vector bundle (see for example [L, §6.2 and 8.1]). In the definition below we use the well-known notation $\binom{\ell}{m}=\frac{\ell(\ell-1)\dots(\ell-m+1)}{m!}$ for $m,\ell\in\mathbb{Z},m\ge 1$.

Definition 3.1. Let X be a smooth projective variety of dimension n. A *$\mathbb{Q}$-twisted vector bundle $\mathcal{E}\langle\delta\rangle$* on X is a pair $(\mathcal{E},\delta)$ where $\mathcal{E}$ is a rank r bundle on X and $\delta\in N^1(X)_{\mathbb{Q}}$.

The Chern classes of $\mathcal{E}\langle\delta\rangle$ are

$$c_i(\mathcal{E}\langle\delta\rangle)=\sum_{k=0}^{i}\binom{r-k}{i-k}c_k(\mathcal{E})\delta^{i-k}\in H^{2i}(X;\mathbb{Q})$$

where we view $\delta\in H^2(X;\mathbb{Q})$.

The Chern character of $\mathcal{E}\langle\delta\rangle$ is $ch(\mathcal{E}\langle\delta\rangle)=ch(\mathcal{E})e^{\delta}$. For $0\le k\le n$, we denote by $ch_k(\mathcal{E}\langle\delta\rangle)$ its degree k part.

Remark 3.2. The definition of Chern classes of $\mathcal{E}\langle\delta\rangle$ [L, Def. 8.1.1] comes of course from the usual formula for $c_i(\mathcal{E}\otimes L)$, where L is a line bundle. Moreover, as in the case of a line bundle, we observe that $c_i(\mathcal{E}\langle\delta\rangle)=0$ when $i>r$: Indeed

$$c_i(\mathcal{E}\langle\delta\rangle)=\sum_{k=0}^{r}\binom{r-k}{i-k}c_k(\mathcal{E})\delta^{i-k}+\sum_{k=r+1}^{i}\binom{r-k}{i-k}c_k(\mathcal{E})\delta^{i-k}.$$

Now, in the second sum we have that $c_k(\mathcal{E})=0$, while in the first sum we have that

$$\binom{r-k}{i-k}=\frac{(r-k)(r-k-1)\dots(r-i+1)}{(i-k)!}=0.$$

We now give a simple result about Newton power sums of $\mathbb{Q}$-twisted vector bundles.

Definition 3.3. Let X be a smooth irreducible variety and let $\mathcal{E}\langle\delta\rangle$ be a $\mathbb{Q}$-twisted vector bundle on X. For each $k\ge 0$, *the k-th Newton power sum of $\mathcal{E}\langle\delta\rangle$* is $p_k(\mathcal{E}\langle\delta\rangle)=k!\,\mathrm{ch}_k(\mathcal{E}\langle\delta\rangle)$.

We have the following

Lemma 3.4. *Let X be a smooth irreducible variety and let $\mathcal{E}\langle\delta\rangle$ be a rank r $\mathbb{Q}$-twisted vector bundle on X. Then, for every $k \geq 1$,*

$$k\, c_k(\mathcal{E}\langle\delta\rangle) = \sum_{i=1}^{k} (-1)^{i-1} p_i(\mathcal{E}\langle\delta\rangle) c_{k-i}(\mathcal{E}\langle\delta\rangle) \tag{3.1}$$

holds in $H^{2k}(X;\mathbb{Q})$.

Proof. By the splitting principle, it is enough to prove the identity after pulling back to a space on which $\mathcal{E}$ splits as a direct sum of line bundles. If $x_1, \dots, x_r$ are the Chern roots of $\mathcal{E}$, then, as in [L, Proof of Lemma 8.1.2], the Chern roots of $\mathcal{E}\langle\delta\rangle$ are $y_i := x_i + \delta, 1 \leq i \leq r$. Therefore

$$c_k(\mathcal{E}\langle\delta\rangle) = e_k(y_1, \dots, y_r)$$

where e_k is the k-th elementary symmetric polynomial, and

$$p_i(\mathcal{E}\langle\delta\rangle) = y_1^i + \cdots + y_r^i.$$

Indeed,

$$\mathrm{ch}(\mathcal{E}\langle\delta\rangle) = ch(\mathcal{E})e^{\delta} = \big(\sum_{j=1}^{r} e^{x_j}\big)e^{\delta} = \sum_{j=1}^{r} e^{y_j} = \sum_{j=1}^{r}\sum_{i\geq 0} \frac{y_j^i}{i!}$$

so that $p_i(\mathcal{E}\langle\delta\rangle) = i!\,\mathrm{ch}_i(\mathcal{E}\langle\delta\rangle) = \sum_{j=1}^{r} y_j^i$. Now set

$$C(t) = \prod_{j=1}^{r}(1 + y_j t) = \sum_{k\geq 0} c_k(\mathcal{E}\langle\delta\rangle)t^k$$

where the sum is clearly a finite sum. Then

$$\frac{C'(t)}{C(t)} = \sum_{j=1}^{r} \frac{y_j}{1 + y_j t}.$$

Expanding each summand and observing that all sums below are finite, gives

$$\frac{y_j}{1 + y_j t} = \sum_{a\geq 0} (-1)^a y_j^{a+1} t^a.$$

Hence

$$\frac{C'(t)}{C(t)} = \sum_{a\geq 0}(-1)^a \left(\sum_{j=1}^{r} y_j^{a+1}\right) t^a = \sum_{i\geq 1}(-1)^{i-1} p_i(\mathcal{E}\langle\delta\rangle)t^{i-1}.$$

Multiplying by $tC(t)$, we obtain

$$tC'(t) = C(t)\sum_{i\geq 1}(-1)^{i-1}p_i(\mathcal{E}\langle\delta\rangle)t^i.$$

Now

$$tC'(t) = \sum_{k\geq 1} k\, c_k(\mathcal{E}\langle\delta\rangle)t^k$$

while

$$C(t)\sum_{i\geq 1}(-1)^{i-1}p_i(\mathcal{E}\langle\delta\rangle)t^i = \left(\sum_{a\geq 0} c_a(\mathcal{E}\langle\delta\rangle)t^a\right)\left(\sum_{i\geq 1}(-1)^{i-1}p_i(\mathcal{E}\langle\delta\rangle)t^i\right).$$

Comparing the coefficient of t^k, we get

$$k\, c_k(\mathcal{E}\langle\delta\rangle) = \sum_{i=1}^{k}(-1)^{i-1}p_i(\mathcal{E}\langle\delta\rangle)c_{k-i}(\mathcal{E}\langle\delta\rangle).$$

□

4. Proof of Theorem 1

We first calculate the Chern character of an Ulrich vector bundle on a hypersurface.

Lemma 4.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of dimension $n \geq 3$ and degree $d \geq 3$. Assume that either n is odd, or n is even and X is very general. Let $\mathcal{E}$ be a rank r Ulrich bundle on X. Then, if $H \in A(X) \otimes \mathbb{Q}$ is the class of a hyperplane section, we have*

$$\mathrm{ch}(\mathcal{E}) = \frac{rd}{1 + e^{-H} + \ldots + e^{-(d-1)H}}.$$

Proof. Let $i : X \hookrightarrow \mathbb{P}^{n+1}$ be the inclusion. Let $h \in A(\mathbb{P}^{n+1}) \otimes \mathbb{Q}$ be the class of a hyperplane, so that $i^*h = H$. As is well known (see for example [B, Prop. 2.1(ii)]), the sheaf $i_*\mathcal{E}$ has a linear resolution on $\mathbb{P}^{n+1}$ of the form

$$0 \to \mathcal{O}_{\mathbb{P}^{n+1}}(-1)^{\oplus(rd)} \to \mathcal{O}_{\mathbb{P}^{n+1}}^{\oplus(rd)} \to i_*\mathcal{E} \to 0$$

hence $\mathrm{ch}(i_*\mathcal{E}) = rd(1 - e^{-h}) \in A(\mathbb{P}^{n+1}) \otimes \mathbb{Q}$. Using the latter and Grothendieck–Riemann–Roch's theorem for a closed embedding [F, §15.2, page 288], we find that

$$rd(1 - e^{-h}) = \mathrm{ch}(i_*\mathcal{E}) = i_*\left(\mathrm{td}(\mathcal{O}_X(dH))^{-1} \cdot \mathrm{ch}(\mathcal{E})\right) = i_*\left(\frac{1 - e^{-dH}}{dH} \cdot \mathrm{ch}(\mathcal{E})\right). \tag{4.1}$$

Under the given hypotheses on X, we have that $i^* : A^k(\mathbb{P}^{n+1}) \otimes \mathbb{Q} \cong \mathbb{Q}h^k \to A^k(X) \otimes \mathbb{Q} \cong \mathbb{Q}H^k$ is an isomorphism for $k \neq \frac{n}{2}$ (in particular for odd n), while for even n and $k = \frac{n}{2}$, any algebraic class in $A^{\frac{n}{2}}(X)$ is of type aH^i (see for example [LR2, Lemma 4.1]), hence in particular so is $\mathrm{ch}_{\frac{n}{2}}(\mathcal{E})$. Therefore, there is $\beta \in A(\mathbb{P}^{n+1}) \otimes \mathbb{Q}$ such that $\mathrm{ch}(\mathcal{E}) = i^*\beta$. Setting $\alpha = \frac{1-e^{-dh}}{dh}$, we have that $i^*\alpha = \frac{1-e^{-dH}}{dH}$. Therefore (4.1) becomes, using $i_*(1) = [X] = dh$ and the push-pull formula,

$$rd(1 - e^{-h}) = i_*\left(i^*\alpha \cdot i^*\beta\right) = i_*\left(1 \cdot i^*(\alpha \cdot \beta)\right) = dh\alpha\beta = (1 - e^{-dh})\beta.$$

Therefore $\beta = \frac{rd(1-e^{-h})}{1-e^{-dh}}$ and then

$$\mathrm{ch}(\mathcal{E}) = i^*\beta = \frac{rd(1 - e^{-H})}{1 - e^{-dH}} = \frac{rd}{1 + e^{-H} + \ldots + e^{-(d-1)H}}.$$

□

We are now ready for the proof of Theorem 1.

Proof. Suppose to begin with that either n is odd, or n is even and X is very general, so that Lemma 4.1 applies. Now, assume that there is a rank r Ulrich bundle $\mathcal{E}$ on X, set $u = \frac{d-1}{2}$ and set

$$\mathcal{F} = \mathcal{E}\langle -uH \rangle.$$

Then, applying Lemma 4.1, we have

$$\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{E})e^{-uH} = \frac{rde^{-uH}}{1 + e^{-H} + \ldots + e^{-2uH}}$$

hence

$$\mathrm{ch}(\mathcal{F}) = \frac{rd}{e^{uH} + e^{(u-1)H} + \ldots + e^{-uH}}. \tag{4.2}$$

Now observe that the identity

$$d = \left(e^{uz} + \ldots + e^{-uz}\right)\left(1 + \sum_{j \geq 1}(-1)^j b_j \frac{z^{2j}}{(2j)!}\right)$$

holds by Lemma 2.3, hence we have that the identity

$$d = \left(e^{uH} + \ldots + e^{-uH}\right)\left(1 + \sum_{j \geq 1}(-1)^j b_j \frac{H^{2j}}{(2j)!}\right)$$

also holds as product of formal power series. Therefore, setting $b_0 = 1$, (4.2) gives that

$$\mathrm{ch}(\mathcal{F}) = \sum_{j \geq 0} (-1)^j r b_j \frac{H^{2j}}{(2j)!}$$

and then we see that, for all $j \geq 0$,

$$p_{2j+1}(\mathcal{F}) = 0, \quad p_{2j}(\mathcal{F}) = (-1)^j r b_j H^{2j}. \tag{4.3}$$

For $0 \leq k \leq n$, define the intersection numbers

$$I_k = c_k(\mathcal{F}) H^{n-k}$$

so that, in particular, $I_0 = H^n = d$. Setting $k = 2s$ in (3.1), intersecting with H^{n-2s} and using (4.3), we find, for all $0 \leq s \leq \frac{n}{2}$, that

$$2sI_{2s} = \sum_{i=1}^{2s} (-1)^{i-1} p_i(\mathcal{F}) c_{2s-i}(\mathcal{F}) H^{n-2s} = \sum_{j=1}^{s} (-1)^{j+1} r b_j I_{2s-2j}. \tag{4.4}$$

We shall prove the following purely numerical fact.

Claim 4.2. *Let $m \in \mathbb{Z}$ be such that $3 \leq m \leq \frac{n}{2}$ and assume that $r < F(m)$. Then*

$$(-1)^{m-1} I_{2m} > 0.$$

Proof. For all $0 \leq s \leq \frac{n}{2}$, set

$$J_s = \frac{I_{2s}}{d} \tag{4.5}$$

so that $J_0 = 1$ and (4.4) becomes

$$2sJ_s = \sum_{j=1}^{s} (-1)^{j+1} r b_j J_{s-j}$$

that is

$$sJ_s = \sum_{j=1}^{s} j A_j J_{s-j}, \tag{4.6}$$

where

$$A_j := (-1)^{j+1} \frac{r b_j}{2j}. \tag{4.7}$$

We will now prove that

$$J_s = \sum_{\lambda \vdash s} \left(\prod_{j \geq 1} \frac{A_j^{s_j}}{s_j!} \right) \tag{4.8}$$

where $\lambda = (1^{s_1} 2^{s_2} \cdots)$ runs over all partitions of $s \geq 0$ such that

$$\sum_{j \geq 1} j s_j = s.$$

First, note that the product on the right-hand side of (4.8) is finite. To see (4.8), define, for every $s \geq 0$,

$$R_0 = 1 \text{ and } R_s = \sum_{\lambda \vdash s} \left(\prod_{j \geq 1} \frac{A_j^{s_j}}{s_j!} \right).$$

Consider the generating function

$$R(t) = \sum_{s \geq 0} R_s t^s.$$

We have

$$R(t)=\sum_{s\geq 0}\sum_{\substack{s_1\geq 0,s_2\geq 0,\dots\\ \sum_{j\geq 1} js_j=s}}\left(\prod_{j\geq 1}\frac{A_j^{s_j}}{s_j!}t^s\right)=\sum_{\substack{(s_1,s_2,\dots),s_i\geq 0\\ \text{with finite support}}}\left(\prod_{j\geq 1}\frac{(A_jt^j)^{s_j}}{s_j!}\right). \tag{4.9}$$

Consider the infinite product of formal power series $\prod_{j\geq 1}(1+F_j(t))$ where

$$F_j(t)=\sum_{s_j\geq 1}\frac{(A_jt^j)^{s_j}}{s_j!}.$$

Then $F_j(0)=0$ and $\deg F_j(t)=j$ since $A_j\neq 0$ by (4.7) and (2.4). Hence

$$\lim_{j\to\infty}\deg F_j(t)=\infty$$

and $\prod_{j\geq 1}(1+F_j(t))$ converges, by [S, Prop. 1.1.9], to a formal power series whose coefficient of t^s is clearly the same as the one in (4.9). This implies that, as formal power series,

$$\sum_{\substack{(s_1,s_2,\dots),s_i\geq 0\\ \text{with finite support}}}\left(\prod_{j\geq 1}\frac{(A_jt^j)^{s_j}}{s_j!}\right)=\prod_{j\geq 1}\left(\sum_{s_j\geq 0}\frac{(A_jt^j)^{s_j}}{s_j!}\right)=\prod_{j\geq 1}e^{A_jt^j}=e^{\sum\limits_{j\geq 1}A_jt^j}$$

where the last step is possible since if $H(t)=\sum_{j\geq 1}A_jt^j$, then $H(0)=0$, hence $e^{H(t)}$ is a well-defined composition of two series [S, §1.1]. Then

$$R(t)=e^{\sum\limits_{j\geq 1}A_jt^j}$$

and therefore

$$R'(t)=R(t)\sum_{j\geq 1}jA_jt^{j-1}$$

and

$$tR'(t)=R(t)\sum_{j\geq 1}jA_jt^j.$$

Comparing the coefficient of t^s, we have, on the left-hand side sR_s and, on the right-hand side,

$$\sum_{\substack{k+j=s\\ k\geq 0,j\geq 1}}R_kjA_j=\sum_{j=1}^{s}jA_jR_{s-j}.$$

Thus, R_s satisfies the same recursion (4.6) and initial value as J_s and therefore $R_s=J_s$, that is (4.8) holds.

Next, consider (4.8) for $s=m$, that is

$$J_m=\sum_{\lambda\vdash m}\prod_{j\geq 1}\frac{A_j^{m_j}}{m_j!}. \tag{4.10}$$

The term corresponding to the partition $\lambda=(1^0,2^0,\dots,m^1)$ in the above is, by (4.7),

$$A_m=(-1)^{m+1}\frac{rb_m}{2m}.$$

Let

$$M:=\frac{rb_m}{2m}$$

so that, using (2.4),

$$(-1)^{m-1}A_m=M>0. \tag{4.11}$$

We shall prove that the sum of the absolute values of all remaining terms in (4.10) is strictly smaller than M.

To this end, let $\ell(\lambda) = \sum_{j\geq 1} m_j$ be the length of a partition λ. For $\ell \geq 2$, let

$$T_\ell = \sum_{\substack{\lambda\vdash m\\ \ell(\lambda)=\ell}} \prod_{j\geq 1} \frac{|A_j|^{m_j}}{m_j!}$$

where we note that sum and product are finite. We get from (4.7) and (2.4) that

$$|A_j| = \frac{rb_j}{2j} = r((2j-1)!)\frac{b_j}{(2j)!} \leq r((2j-1)!)\frac{2d\sin(\frac{\pi}{d})}{\pi}q^j \tag{4.12}$$

and

$$M = |A_m| = r((2m-1)!)\frac{b_m}{(2m)!} \geq r((2m-1)!)\frac{5}{4}q^m. \tag{4.13}$$

Now define

$$P_{m,\ell} = \sum_{\substack{\lambda\vdash m\\ \ell(\lambda)=\ell}} \prod_{j\geq 1} \frac{((2j-1)!)^{m_j}}{m_j!}$$

where we note that sum and product are finite. Using (4.12) and (4.13) we get

$$\begin{aligned}\frac{T_\ell}{M} &= \sum_{\substack{\lambda\vdash m\\ \ell(\lambda)=\ell}} \frac{1}{M}\prod_{j\geq 1}\frac{|A_j|^{m_j}}{m_j!} \leq \sum_{\substack{\lambda\vdash m\\ \ell(\lambda)=\ell}} \frac{1}{M}\prod_{j\geq 1}\frac{r^{m_j}((2j-1)!)^{m_j}\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^{m_j} q^{jm_j}}{m_j!} = \\ &= \frac{r^\ell q^m\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^{\ell}}{M}\sum_{\substack{\lambda\vdash m\\ \ell(\lambda)=\ell}}\prod_{j\geq 1}\frac{((2j-1)!)^{m_j}}{m_j!} \leq \frac{r^{\ell-1}\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^{\ell}}{\frac{5}{4}(2m-1)!}P_{m,\ell}.\end{aligned} \tag{4.14}$$

We now estimate $P_{m,\ell}$. Consider, for $\ell \leq m$,

$$Q_{m,\ell} := \frac{1}{\ell!}\sum_{\substack{i_1+\cdots+i_\ell=m\\ i_a\geq 1}}\prod_{a=1}^{\ell}(2i_a-1)!.$$

We will prove that $P_{m,\ell} = Q_{m,\ell}$. To this end, for every ℓ-tuple $(i_1,\dots,i_\ell)$ such that $i_a \geq 1$ for $a \geq 1$ and $i_1+\cdots+i_\ell = m$, define, for every $j \geq 1$,

$$m_j = \#\{a \in \{1,\dots,\ell\} : i_a = j\}$$

so that $\sum\limits_{j\geq 1} m_j = \ell$, $\sum\limits_{j\geq 1} jm_j = m$ and

$$\prod_{a=1}^{\ell}(2i_a-1)! = \prod_{j\geq 1}((2j-1)!)^{m_j}.$$

Vice versa, for every partition $(1^{m_1}2^{m_2}\cdots)$ of m of length ℓ, there are $\frac{\ell!}{\prod_{j\geq 1} m_j!}$ possibilities for $i_1 + \cdots + i_\ell = m$ with all $i_a \geq 1$. Hence

$$Q_{m,\ell} = \frac{1}{\ell!}\sum_{\substack{i_1+\cdots+i_\ell=m\\ i_a\geq 1}}\prod_{a=1}^{\ell}(2i_a-1)! = \frac{1}{\ell!}\sum_{\substack{\sum_j jm_j=m\\ \sum_j m_j=\ell}}\frac{\ell!}{\prod_{j\geq 1}m_j!}\prod_{j\geq 1}((2j-1)!)^{m_j} = P_{m,\ell}.$$

For positive integers a, b, one has

$$a!b! \leq (a+b-1)!.$$

Applying this repeatedly gives

$$\prod_{a=1}^{\ell}(2i_a-1)! \leq (2m-2\ell+1)!.$$

Therefore

$$P_{m,\ell} = Q_{m,\ell} = \frac{1}{\ell!} \sum_{\substack{i_1+\cdots+i_\ell=m \\ i_a\geq 1}} \prod_{a=1}^{\ell} (2i_a-1)! \leq \frac{1}{\ell!} \sum_{\substack{i_1+\cdots+i_\ell=m \\ i_a\geq 1}} (2m-2\ell+1)! = \frac{1}{\ell!}\binom{m-1}{\ell-1}(2m-2\ell+1)!.$$

Now, using (4.14), we deduce that

$$\frac{T_\ell}{M} \leq \frac{r^{\ell-1}\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^\ell}{\frac{5}{4}(2m-1)!}\frac{1}{\ell!}\binom{m-1}{\ell-1}(2m-2\ell+1)!.$$

Define the right-hand side above to be

$$U_\ell = \frac{r^{\ell-1}\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^\ell}{\frac{5}{4}(2m-1)!}\frac{1}{\ell!}\binom{m-1}{\ell-1}(2m-2\ell+1)!.$$

Then

$$\frac{T_\ell}{M} \leq U_\ell. \tag{4.15}$$

For $\ell = 2$, we compute

$$U_2 = \frac{r\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^2}{5(2m-1)} \tag{4.16}$$

while, for $2 \leq l \leq m-1$,

$$\frac{U_{\ell+1}}{U_\ell} = \frac{r\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)}{2\ell(\ell+1)(2m-2\ell+1)}.$$

Since $\ell(\ell+1)(2m-2\ell+1) \geq 6(2m-3)$ for $2 \leq \ell \leq m-1$, we get that

$$\frac{U_{\ell+1}}{U_\ell} \leq \frac{rd\sin(\frac{\pi}{d})}{6\pi(2m-3)}. \tag{4.17}$$

Set

$$x = \frac{6\pi(2m-3)}{rd\sin(\frac{\pi}{d})}$$

so that (4.17) gives

$$\frac{U_{\ell+1}}{U_\ell} \leq \frac{1}{x}. \tag{4.18}$$

Observe that $x > 1$: since $r < F(m)$ we have in particular that $r < 2m \leq \frac{6\pi(2m-3)}{d\sin(\frac{\pi}{d})}$. Combining (4.18) with (4.15) we have

$$\sum_{\ell=2}^{m}\frac{T_\ell}{M} \leq \sum_{\ell=2}^{m} U_\ell \leq U_2\sum_{\ell=2}^{m}\frac{1}{x^{\ell-2}} \leq U_2\sum_{k\geq 0}\frac{1}{x^k} = \frac{U_2 x}{x-1} < 1 \tag{4.19}$$

the latter because we have from (4.16) that

$$U_2 = \frac{r\left(\frac{2d\sin(\frac{\pi}{d})}{\pi}\right)^2}{5(2m-1)} < \frac{x-1}{x} = \frac{6\pi(2m-3)-rd\sin(\frac{\pi}{d})}{6\pi(2m-3)}$$

if and only if

$$r < \frac{30\pi^2(2m-1)(2m-3)}{24d^2(2m-3)\sin^2(\frac{\pi}{d}) + 5\pi d(2m-1)\sin(\frac{\pi}{d})} = B(m)$$

and the above holds by the hypothesis $r < F(m)$. It follows from (4.19) that

$$\sum_{\ell=2}^{m} T_\ell < M.$$

In the expansion (4.10),

$$J_m = \sum_{\lambda \vdash m} \prod_{j \geq 1} \frac{A_j^{m_j}}{m_j!} = A_m + \sum_{\substack{\lambda \vdash m \\ \ell(\lambda) \geq 2}} \prod_{j \geq 1} \frac{A_j^{m_j}}{m_j!} = A_m + \sum_{\ell \geq 2} \sum_{\substack{\lambda \vdash m \\ \ell(\lambda) = \ell}} \prod_{j \geq 1} \frac{A_j^{m_j}}{m_j!}$$

we have that $|A_m| = M > 0$, while

$$|\sum_{\ell \geq 2} \sum_{\substack{\lambda \vdash m \\ \ell(\lambda) = \ell}} \prod_{j \geq 1} \frac{A_j^{m_j}}{m_j!}| \leq \sum_{\ell \geq 2} \sum_{\substack{\lambda \vdash m \\ \ell(\lambda) = \ell}} \prod_{j \geq 1} \frac{|A_j|^{m_j}}{m_j!} = \sum_{\ell=2}^{m} T_\ell < M$$

hence J_m has the same sign as A_m and (4.11) gives that $(-1)^{m-1} J_m > 0$, hence $(-1)^{m-1} I_{2m} > 0$ by (4.5). This proves Claim 4.2. □

To finish the proof of Theorem 1, we choose $m = \lfloor \frac{n}{2} \rfloor$ in Claim 4.2. Suppose that $r < F(m)$. Then $c_{2m}(\mathcal{F}) H^{n-2m} = I_{2m} \neq 0$ by Claim 4.2. Since $r < 2m$, this is a contradiction (see Remark 3.2) and the theorem is proved for n odd or n even and X very general. Finally suppose that n is even and X has an Ulrich bundle $\mathcal{E}$ of rank r. Then its hyperplane section has odd dimension $n-1$ and again an Ulrich bundle, restriction of $\mathcal{E}$, of rank r. Hence, from the odd case, we get that $r \geq F(\frac{n-2}{2})$ and the proof is complete. □

5. Numerical values

We show here some calculations that allow to understand when $F(m) = 2m$.

Using the fact that $\sin(x) \leq x$ for $x \geq 0$ it is easily shown that $F(m) = 2m$, that is $2m \leq B(m)$, for $m \geq 21$. We also note that $B(m) \geq \frac{3m}{2}$ for $3 \leq m \leq 20$.

For $3 \leq m \leq 20$, setting $x = \frac{\sin(\frac{\pi}{d})}{\frac{\pi}{d}}$, the inequality $2m \leq B(m)$ is equivalent to

$$48m(2m-3)x^2 + 10m(2m-1)x - 30(2m-1)(2m-3) \leq 0$$

that is

$$0 \leq x \leq \frac{-5m(2m-1) + \sqrt{5m(2m-1)(2592 - 3461m + 1162m^2)}}{48m(2m-3)}.$$

Setting

$$a(m) = \frac{-5m(2m-1) + \sqrt{5m(2m-1)(2592 - 3461m + 1162m^2)}}{48m(2m-3)}$$

we see that

$$a(m)y - \sin(y) \geq 0$$

for $0 \leq y \leq \frac{\pi}{2}$ if and only if $y_0(m) \leq y \leq \frac{\pi}{2}$, where $y_0(m)$ is the only solution of the equation $a(m)y - \sin(y) = 0$ for $0 < y \leq \frac{\pi}{2}$.

It follows that, for $3 \leq m \leq 20$, we have that $F(m) = 2m$ if and only if $d \leq \frac{\pi}{y_0(m)}$. In Table 1 below we give the values of $y_0(m)$ and d such that $F(m) = 2m$.

TABLE 1. Values of d for which $F(m) = 2m$

m	$y_0(m)$	value
3	0.93	$d = 3$
4	0.74	$d \leq 4$
5	0.62	$d \leq 4$
6	0.54	$d \leq 5$
7	0.48	$d \leq 6$
8	0.43	$d \leq 7$
9	0.38	$d \leq 8$
10	0.35	$d \leq 8$
11	0.31	$d \leq 9$
12	0.28	$d \leq 10$
13	0.25	$d \leq 12$
14	0.23	$d \leq 13$
15	0.20	$d \leq 15$
16	0.17	$d \leq 17$
17	0.15	$d \leq 20$
18	0.12	$d \leq 25$
19	0.09	$d \leq 34$
20	0.04	$d \leq 65$

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